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Blow-up phenomena for scalar-flat metrics on manifolds with boundary

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ABSTRACT

Let (M^n, g) be a compact Riemannian manifold with boundary ∂M. This article is concerned with the set of scalar-flat metrics on M which are in the conformal class of g and have ∂M as a constant mean curvature hypersurface. We construct examples of metrics on the unit ball B^n, in dimensions $n \geqslant 25$, for which this set is non-compact. These manifolds have umbilic boundary, but they are not conformally equivalent to B^n.

1. Introduction

Let (M^n, g) be a compact Riemannian manifold with boundary ∂M and dimension $n \geqslant 3$. In 1992, J. Escobar addressed the question of finding a scalar-flat conformal metric $\tilde{g} = u^{\frac{4}{n-2}} g$ which has ∂M as a constant mean curvature hypersurface. This problem was studied in [2,9,16–18,27,28]. In analytical terms, it corresponds to the existence of a positive solution to the equations

$$\begin{cases} \Delta_g u - c_n R_g u = 0, & \text{in } M, \\ \dfrac{\partial u}{\partial \eta} - d_n \kappa_g u + K u^{\frac{n}{n-2}} = 0, & \text{on } \partial M, \end{cases} \tag{1.1}$$

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for some constant K, where $c_n = \frac{n-2}{4(n-1)}$ and $d_n = \frac{n-2}{2}$. Here, Δ_g is the Laplace–Beltrami operator, R_g is the scalar curvature, κ_g is the mean curvature of ∂M and η is the inward unit normal vector to ∂M.

Escobar's question was motivated by the classical Yamabe problem, which consists of finding a conformal metric of constant scalar curvature on a given closed Riemannian manifold. This was completely solved after the works of H. Yamabe [35], N. Trudinger [34], T. Aubin [4] and R. Schoen [30]. (See [22] and [32] for nice surveys on the issue.) Conformal metrics of constant scalar curvature and zero boundary mean curvature on the boundary were studied in [7,15] (see also [3] and [20]).

The solutions to Eqs. (1.1) are the critical points of the functional

$$Q(u) = \frac{\int_M |du|_g^2 + c_n R_g u^2 \, dv_g + \int_{\partial M} d_n \kappa_g u^2 \, d\sigma_g}{\left(\int_{\partial M} |u|^{\frac{2(n-1)}{n-2}} \, d\sigma_g\right)^{\frac{n-2}{n-1}}},$$

where dv_g and $d\sigma_g$ denote the volume forms of M and ∂M, respectively. In order to prove the existence of these solutions, Escobar introduced the conformally invariant Sobolev quotient

$$Q(M, \partial M) = \inf\left\{Q(u);\ u \in C^1(\bar{M}),\ u \not\equiv 0 \text{ on } \partial M\right\}.$$

In this work we are interested in the question of whether the full set of solutions to (1.1) is compact. A necessary condition is that M is not conformally equivalent to the standard ball B^n. We point out that if Eqs. (1.1) have a solution $u > 0$ with K positive (resp. zero and negative), then $Q(M, \partial M)$ has to be positive (resp. zero and negative). If $K < 0$, the solution to Eqs. (1.1) is unique. If $K = 0$, Eqs. (1.1) become linear and the solutions are unique up to a multiplication by a positive constant. Hence, the only interesting case is the one when $K > 0$.

The problem of compactness of solutions to Eqs. (1.1) was studied by V. Felli and M. Ould Ahmedou in the conformally flat case with umbilic boundary [18] and in the three-dimensional case with umbilic boundary [19]. In [1], the author proved compactness for dimensions $n \geqslant 7$ under a generic condition. Other compactness results for similar equations were obtained by Z. Djadli, A. Malchiodi and M. Ould Ahmedou in [11,12], by Z. Han and Y. Li in [20] and by M. Ould Ahmedou in [29].

In the case of manifolds without boundary, the question of compactness of the full set of solutions to the Yamabe equation was first raised by R. Schoen in a topics course at Stanford University in 1988. A necessary condition is that the manifold M^n is not conformally equivalent to the sphere S^n. This problem was studied in [13,14,23–26,31,33] and was completely solved in a series of three papers: [6,8] and [21]. In [6], S. Brendle discovered the first smooth counterexamples for dimensions $n \geqslant 52$ (see [5] for nonsmooth examples). In [21], M. Khuri, F. Marques and R. Schoen proved compactness for dimensions $3 \leqslant n \leqslant 24$. Finally, in [8], Brendle and Marques extended the counterexamples of [6] to the remaining dimensions $25 \leqslant n \leqslant 51$.

It is expected that, as in the case of manifolds without boundary, there should be a critical dimension n_0 such that compactness in the case of manifolds with boundary holds for $n < n_0$ and fails for $n \geqslant n_0$. In this work we partially answer this question by showing that compactness fails for dimensions $n \geqslant 25$. More precisely we prove:

Main Theorem. *Let $n \geqslant 25$. Then there exist a smooth Riemannian metric g on B^n and a sequence of positive smooth functions $\{v_\nu\}_{\nu=1}^{\infty}$ with the following properties:*

(i) *g is not conformally flat;*
(ii) *∂B^n is umbilic with respect to the induced metric by g;*
(iii) *for all ν, v_ν is a solution to Eqs.* (1.1) *with a constant $K > 0$ and $M = B^n$;*
(iv) $Q(v_\nu) < Q(B^n, \partial B)$ *for all ν;*
(v) $\sup_{\partial B^n} v_\nu \to \infty$ *as $\nu \to \infty$.*

In order to prove the Main Theorem, we follow the program adopted in [6] and [8]. In Section 2, we show that the problem can be reduced to finding critical points of a certain function $\mathcal{F}_g(\xi, \epsilon)$, where ξ is a vector in $\mathbb{R}^{n-1}$ and ϵ is a positive real number. In Section 3, we show that the function $\mathcal{F}_g(\xi,\epsilon)$ can be approximated by an auxiliary function $F(\xi,\epsilon)$. In Section 4, we prove that the function $F(\xi,\epsilon)$ has a strict local minimum point. The cases $n \geqslant 53$ and $25 \leqslant n \leqslant 52$ are handled separately in Sections 4.1 and 4.2 respectively. Finally, in Section 5, we use a perturbation argument to construct critical points of the function $\mathcal{F}_g(\xi,\epsilon)$ and prove the non-compactness theorem.

Notation. Throughout this work we will make use of the index notation for tensors. We will adopt the summation convention whenever confusion is not possible and use indices $1 \leqslant i, i, j, k, l, m, p, q, r \leqslant n-1$ and $1 \leqslant a, b, c, d \leqslant n$. We also define constants $c_n = \frac{n-2}{4(n-1)}$ and $d_n = \frac{n-2}{2}$.

We will denote by Δ_g the Laplace–Beltrami operator. The volume forms of M and ∂M will be denoted by dv_g and $d\sigma_g$, respectively. By η we will denote the inward unit normal vector to ∂M. The scalar curvature will be denoted by R_g, the second fundamental form of ∂M by π_{kl} and the mean curvature, $\frac{1}{n-1} tr(\pi_{kl})$, by κ_g.

By $\mathbb{R}^n_+$ we will denote the half-space $\{x = (x_1, \dots, x_n) \in \mathbb{R}^n;\ x_n \geqslant 0\}$. If $x \in \mathbb{R}^n_+$ we set $\bar{x} = (x_1, \dots, x_{n-1}, 0) \in \partial\mathbb{R}^n_+ \cong \mathbb{R}^{n-1}$. For any $x_0 \in \mathbb{R}^n_+$ we set $B^+_r(x_0) = \{x \in \mathbb{R}^n_+;\ |x - x_0| < r\}$. The n-dimensional sphere of radius r in $\mathbb{R}^{n+1}$ will be denoted by S^n_r and σ_n will denote the area of the n-dimensional unit sphere S^n_1.

2. Lyapunov–Schmidt reduction

Given a pair $(\xi,\epsilon) \in \mathbb{R}^{n-1} \times (0,\infty)$ we set

$$u_{(\xi,\epsilon)}(x) = \left(\frac{\epsilon}{(\epsilon + x_n)^2 + |\bar{x} - \xi|^2}\right)^{\frac{n-2}{2}}, \quad \text{for } x \in \mathbb{R}^n_+.$$

Observe that $u_{(\xi,\epsilon)}$ satisfies

$$\begin{cases} \Delta u_{(\xi,\epsilon)} = 0, & \text{in } \mathbb{R}^n_+, \\ \dfrac{\partial}{\partial x_n} u_{(\xi,\epsilon)} + (n-2) u_{(\xi,\epsilon)}^{\frac{n}{n-2}} = 0, & \text{on } \partial\mathbb{R}^n_+, \end{cases} \tag{2.1}$$

and

$$\int_{\partial\mathbb{R}^n_+} u_{(\xi,\epsilon)}^{\frac{2(n-1)}{n-2}} = \left(\frac{Q(B^n, \partial B)}{n-2}\right)^{n-1}. \tag{2.2}$$

Let us define

$$\phi_{(\xi,\epsilon,n)}(x) = \left(\frac{\epsilon}{(\epsilon + x_n)^2 + |\bar{x} - \xi|^2}\right)^{\frac{n}{2}} \frac{\epsilon^2 - x_n^2 - |\bar{x} - \xi|^2}{(\epsilon + x_n)^2 + |\bar{x} - \xi|^2}$$

and

$$\phi_{(\xi,\epsilon,k)}(x) = \left(\frac{\epsilon}{(\epsilon + x_n)^2 + |\bar{x} - \xi|^2}\right)^{\frac{n}{2}} \frac{2\epsilon(x_k - \xi_k)}{(\epsilon + x_n)^2 + |\bar{x} - \xi|^2}$$

for $x \in \mathbb{R}^n_+$ and $k = 1, \dots, n-1$. Observe that

$$\phi_{(\xi,\epsilon,n)}(x)\cdot\big((\epsilon+x_n)^2+|\bar{x}-\xi|^2\big)=-\frac{2\epsilon^2}{n-2}\frac{\partial}{\partial\epsilon}u_{(\xi,\epsilon)}(x),$$
$$\phi_{(\xi,\epsilon,k)}(x)\cdot\big((\epsilon+x_n)^2+|\bar{x}-\xi|^2\big)=\frac{2\epsilon^2}{n-2}\frac{\partial}{\partial\xi_k}u_{(\xi,\epsilon)}(x),$$

for $k=1,\dots,n-1$, and that $\|\phi_{(\xi,\epsilon,a)}\|_{L^{\frac{2(n-1)}{n}}(\partial\mathbb{R}^n_+)}$ is independent of $(\epsilon,\xi)\in\mathbb{R}^{n-1}\times(0,\infty)$, for any $a=1,\dots,n$.

We also set

$$\Sigma=\Big\{w\in L^{\frac{2n}{n-2}}\big(\mathbb{R}^n_+\big)\cap L^{\frac{2(n-1)}{n-2}}\big(\partial\mathbb{R}^n_+\big)\cap H^1_{\mathrm{loc}}\big(\mathbb{R}^n_+\big);\ \int_{\mathbb{R}^n_+}|dw|^2<\infty\Big\},$$

$$\Sigma_{(\xi,\epsilon)}=\Big\{w\in\Sigma;\ \int_{\partial\mathbb{R}^n_+}\phi_{(\xi,\epsilon,a)}w=0,\ a=1,\dots,n\Big\}$$

and $\|w\|_\Sigma=(\int_{\mathbb{R}^n_+}|dw|^2)^{\frac12}$ for $w\in\Sigma$. Observe that $u_{(\xi,\epsilon)}\in\Sigma_{(\xi,\epsilon)}$ for each $(\xi,\epsilon)\in\mathbb{R}^{n-1}\times(0,\infty)$. By Sobolev's inequality, there exists $C=C(n)>0$ such that

$$\Big(\int_{\mathbb{R}^n_+}|w|^{\frac{2n}{n-2}}\Big)^{\frac{n-2}{n}}+\Big(\int_{\partial\mathbb{R}^n_+}|w|^{\frac{2(n-1)}{n-2}}\Big)^{\frac{n-2}{n-1}}\leqslant C\int_{\mathbb{R}^n_+}|dw|^2 \tag{2.3}$$

for all $w\in\Sigma$.

In what follows in this section we are going to find, for each pair $(\xi,\epsilon)\in\mathbb{R}^{n-1}\times(0,\infty)$, a function $v_{(\xi,\epsilon)}\in\Sigma$ which is an approximate weak solution to a Yamabe-type problem (1.1) on $\mathbb{R}^n_+$. Then we will show that $v_{(\xi,\epsilon)}$ is in fact a classical solution to this problem whenever (ξ,ϵ) is a critical point of a certain energy function defined on $\mathbb{R}^{n-1}\times(0,\infty)$.

Notation. In this section we suppose that g is a Riemannian metric on $\mathbb{R}^n_+$ expressed as $g=\exp(h)$, where h is a trace-free symmetric two-tensor satisfying $h(x)=0$ for any $|x|\geqslant 1$.

Let $B^n=B^n_{1/2}(0,\dots,0,-\frac12)\subset\mathbb{R}^n$ be the ball with radius $\frac12$ and center $(0,\dots,0,-\frac12)$. Let $z_1,\dots,z_n$ be the coordinates of B^n taken with center $(0,\dots,0,-\frac12)$. The properties of the conformal equivalence between B^n and $\mathbb{R}^n_+\cup\{\infty\}$ that we are going to use are established in the next lemma.

Lemma 2.1. *For each pair $(\xi,\epsilon)\in\mathbb{R}^{n-1}\times(0,\infty)$, the expression*

$$\mathcal{C}_{(\xi,\epsilon)}(x)=\frac{\epsilon(x_1-\xi_1,\dots,x_{n-1}-\xi_{n-1},x_n+\epsilon)}{|\bar{x}-\xi|^2+(x_n+\epsilon)^2}+(0,\dots,0,-1)$$

defines a conformal equivalence

$$\mathcal{C}_{(\xi,\epsilon)}:\mathbb{R}^n_+\to B^n\setminus\big\{(0,\dots,0,-1)\big\}$$

that satisfies $\mathcal{C}^*_{(\xi,\epsilon)}\delta_{B^n}=u_{(\xi,\epsilon)}^{\frac{4}{n-2}}\delta$, where δ_{B^n} is the Euclidean metric on B^n and δ is the Euclidean metric on $\mathbb{R}^n_+$. For any smooth function f on $\mathbb{R}^n_+$, we have

$$\Delta_{B^n}\tilde{u}_{(\xi,\epsilon)}=u_{(\xi,\epsilon)}^{-\frac{n+2}{n-2}}\Delta f \tag{2.4}$$

and

$$\frac{\partial}{\partial\eta}\tilde{u}_{(\xi,\epsilon)} - (n-2)\tilde{u}_{(\xi,\epsilon)} = u_{(\xi,\epsilon)}^{-\frac{n}{n-2}}\frac{\partial f}{\partial x_n}, \tag{2.5}$$

where $\tilde{u}_{(\xi,\epsilon)} = (f u_{(\xi,\epsilon)}^{-1}) \circ \mathcal{C}_{(\xi,\epsilon)}^{-1}$. *Moreover,*

$$z_n \circ \mathcal{C}_{(\xi,\epsilon)} = -\frac{\epsilon}{n-2}u_{(\xi,\epsilon)}^{-1}\frac{\partial}{\partial\epsilon}u_{(\xi,\epsilon)} = \frac{1}{2}u_{(\xi,\epsilon)}^{-\frac{n}{n-2}}\phi_{(\xi,\epsilon,n)} \tag{2.6}$$

and

$$z_k \circ \mathcal{C}_{(\xi,\epsilon)} = \frac{\epsilon}{n-2}u_{(\xi,\epsilon)}^{-1}\frac{\partial}{\partial\xi_k}u_{(\xi,\epsilon)} = \frac{1}{2}u_{(\xi,\epsilon)}^{-\frac{n}{n-2}}\phi_{(\xi,\epsilon,k)}, \quad k = 1,\ldots,n-1. \tag{2.7}$$

Proof. These are direct computations. The assertions (2.4) and (2.5) follow from the following properties of the conformal operators $L_g = \Delta_g - c_n R_g$ and $B_g = \frac{\partial}{\partial\eta} - d_n\kappa_g$:

$$L_{u^{\frac{4}{n-2}}g}\big(fu^{-1}\big) = u^{-\frac{n+2}{n-2}}L_g f \quad \text{and} \quad B_{u^{\frac{4}{n-2}}g}\big(fu^{-1}\big) = u^{-\frac{n}{n-2}}B_g f. \qquad \square \tag{2.8}$$

We will also need the following estimate for functions in $H^1(B^n)$.

Lemma 2.2. *There exists $\theta = \theta(n) > 0$ such that*

$$\int_{B^n}|dw|^2 - 2\int_{\partial B^n} w^2 - 2\theta\bigg(\int_{B^n}|dw|^2 + (n-2)\int_{\partial B^n} w^2\bigg) + \frac{4}{\theta}\bigg(\int_{\partial B^n} w\bigg)^2 \geqslant 0$$

for any $w \in H^1(B^n)$ such that $w \perp_{L^2(\partial B^n)} \{z_1,\ldots,z_n\}$. Here, we are following the notations of Lemma 2.1.

Proof. First we fix $0 \not\equiv w \in H^1(B^n)$ such that $w \perp_{L^2(\partial B^n)} \{1, z_1,\ldots,z_n\}$. Since

$$\inf\left\{\frac{\int_{B^n}|d\psi|^2}{\int_{\partial B^n}\psi^2},\ \text{such that } \psi \in H^1\big(B^n\big),\ \psi \not\equiv 0 \text{ on } \partial B^n \text{ and } \psi \perp_{L^2(\partial B^n)} 1\right\} = 2$$

and this infimum is realized only by the functions $z_1,\ldots,z_n$, we see that

$$\int_{B^n}|dw|^2 - 2\int_{\partial B^n} w^2 > 0.$$

Hence,

$$\int_{B^n}|dw|^2 - 2\int_{\partial B^n} w^2 \geqslant 2\theta\bigg(\int_{B^n}|dw|^2 + (n-2)\int_{\partial B^n} w^2\bigg) \tag{2.9}$$

holds for any $\theta > 0$ satisfying

$$\theta \leqslant \theta(w) = \frac{1}{2}\frac{\int_{B^n}|dw|^2 - 2\int_{\partial B^n} w^2}{\int_{B^n}|dw|^2 + (n-2)\int_{\partial B^n} w^2}$$

and the equality is realized by $\theta = \theta(w)$.

We claim that there exists $\theta_0 > 0$ such that $\theta(w) \geqslant \theta_0$ for any $w \in H^1(B^n)$ satisfying $w \perp_{L^2(\partial B^n)} \{1, z_1, \dots, z_n\}$. Suppose by contradiction this is not true. Thus we can choose a sequence $\{w_j\}_{j=1}^{\infty} \subset H^1(B^n)$ such that $w_j \perp_{L^2(\partial B^n)} \{1, z_1, \dots, z_n\}$ and $\theta(w_j) \to 0$ as $j \to \infty$. Hence

$$\int_{B^n} |dw_j|^2 - 2\int_{\partial B^n} w_j^2 = 2\theta(w_j)\biggl(\int_{B^n} |dw_j|^2 + (n-2)\int_{\partial B^n} w_j^2\biggr)$$

holds and we can assume that $\int_{B^n} |dw_j|^2 = 1$ for any j. Thus, $\int_{\partial B^n} w_j^2 \leqslant \frac{1}{2}$ for all j and we can suppose that $w_j \rightharpoonup w_0$ in $H^1(B^n)$ for some w_0. Since $H^1(B^n)$ is compactly embedded in $L^2(\partial B^n)$, we know that $w_0 \perp_{L^2(\partial B^n)} \{1, z_1, \dots, z_n\}$. Let us first assume that $w_0 \not\equiv 0$. We set

$$\beta = \int_{B^n} |dw_0|^2 - 2\int_{\partial B^n} w_0^2 > 0.$$

Since $\liminf_{i\to\infty} \int_{B^n} |dw_j|^2 \geqslant \int_{B^n} |dw_0|^2$ and $\lim_{i\to\infty} \int_{\partial B^n} w_j^2 = \int_{\partial B^n} w^2$, we can assume that

$$\int_{B^n} |dw_j|^2 - 2\int_{\partial B^n} w_j^2 \geqslant \frac{\beta}{2} \quad \text{for all } j.$$

On the other hand,

$$\frac{\beta}{n}\biggl\{\int_{B^n} |dw_j|^2 + (n-2)\int_{\partial B^n} w_j^2\biggr\} \leqslant \frac{\beta}{2},$$

since $\int_{B^n} |dw_j|^2 = 1$ and $\int_{\partial B^n} w_j^2 \leqslant \frac{1}{2}$. Hence,

$$\begin{aligned} &2\theta(w_j)\biggl(\int_{B^n} |dw_j|^2 + (n-2)\int_{\partial B^n} w_j^2\biggr) \\ &\quad = \int_{B^n} |dw_j|^2 - 2\int_{\partial B^n} w_j^2 \geqslant \frac{\beta}{n}\biggl(\int_{B^n} |dw_j|^2 + (n-2)\int_{\partial B^n} w_j^2\biggr), \end{aligned}$$

which implies that $2\theta(w_j) \geqslant \frac{\beta}{n}$ for all j and contradicts the fact that $\theta(w_j) \to 0$.

Thus we must have $w_0 \equiv 0$, which implies that $\int_{\partial B^n} w_j^2 \to 0$ as $j \to \infty$. Then, if we set $\tilde{w}_j = (\int_{\partial B^n} w_j^2)^{-\frac{1}{2}} w_j$, we have $\tilde{w}_j \rightharpoonup \tilde{w}_0$ in $H^1(B^n)$, for some $\tilde{w}_0$. Moreover,

$$0 = \lim_{j\to\infty} \int_{B^n} |d\tilde{w}_j|^2 \geqslant \int_{B^n} |d\tilde{w}_0|^2$$

and

$$\int_{\partial B^n} \tilde{w}_j^2 = 1 = \int_{\partial B^n} \tilde{w}_0^2.$$

From this we conclude that $\tilde{w}_0 \equiv \text{const} \neq 0$, which contradicts the fact that $\tilde{w}_0 \perp_{L^2(\partial B^n)} 1$. This proves that there exists $\theta_0 > 0$ such that $\theta(w) \geqslant \theta_0$ for any $w \in H^1(B^n)$ satisfying $w \perp_{L^2(\partial B^n)} \{1, z_1, \dots, z_n\}$. In particular, (2.9) holds, with $\theta = \theta_0$, for any $w \in H^1(B^n)$ satisfying $w \perp_{L^2(\partial B^n)} \{1, z_1, \dots, z_n\}$.

Now, let $w \in H^1(B^n)$ satisfy $w \perp_{L^2(\partial B^n)} \{z_1, \dots, z_n\}$. We write $w = w_1 + b$ where b is a constant and $w_1 \perp_{L^2(\partial B^n)} 1$. Then we have

$$\begin{aligned}
&\int_{B^n} |dw|^2 - 2\int_{\partial B^n} w^2 - 2\theta_0\Big(\int_{B^n} |dw|^2 + (n-2)\int_{\partial B^n} w^2\Big) + \frac{4}{\theta_0}\Big(\int_{\partial B^n} w\Big)^2 \\
&\quad = \int_{B^n} |dw_1|^2 - 2\int_{\partial B^n} w_1^2 - 2\theta_0\Big(\int_{B^n} |dw_1|^2 + (n-2)\int_{\partial B^n} w_1^2\Big) \\
&\qquad - 2\big(1+(n-2)\theta_0\big)\int_{\partial B^n} b^2 + \frac{4}{\theta_0}\Big(\int_{\partial B^n} b\Big)^2 \\
&\quad \geqslant \Big(\frac{4}{\theta_0} - 2 - 2(n-2)\theta_0\Big)\int_{\partial B^n} b^2.
\end{aligned}$$

Choosing θ_0 smaller if necessary, we can suppose that $\frac{4}{\theta_0} - 2 - 2(n-2)\theta_0 > 0$ and the result follows. □

The proofs of the next four propositions are similar to Propositions 1, 4, 5 and 6 in [6]. Hence, we will just sketch some proofs, pointing out the necessary modifications, and omit others.

Proposition 2.3. *If $|h(x)| + |\partial h(x)| + |\partial^2 h(x)| \leqslant \alpha \leqslant 1$ for any $x \in \mathbb{R}^n_+$, then there exists $C = C(n) > 0$ such that*

$$\|\Delta_g u_{(\xi,\epsilon)} - c_n R_g u_{(\xi,\epsilon)}\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n_+)} + \|d_n \kappa_g u_{(\xi,\epsilon)}\|_{L^{\frac{2(n-1)}{n}}(\partial\mathbb{R}^n_+)} \leqslant C\alpha$$

for all pairs $(\xi,\epsilon) \in \mathbb{R}^{n-1} \times (0,\infty)$.

Proposition 2.4. *There exists $0 < \alpha_0 = \alpha_0(n) \leqslant 1$ such that, whenever $|h(x)| + |\partial h(x)| + |\partial^2 h(x)| \leqslant \alpha_0$ for all $x \in \mathbb{R}^n_+$, the following holds: given any pair $(\xi,\epsilon) \in \mathbb{R}^{n-1} \times (0,\infty)$ and functions $f \in L^{\frac{2n}{n+2}}(\mathbb{R}^n_+)$ and $\bar{f} \in L^{\frac{2(n-1)}{n}}(\partial\mathbb{R}^n_+)$ there exists a unique $w \in \Sigma_{(\xi,\epsilon)}$ such that*

$$\int_{\mathbb{R}^n_+} \big(\langle dw, d\psi\rangle_g + c_n R_g w\psi\big) + \int_{\partial\mathbb{R}^n_+} \big(d_n \kappa_g w\psi - n u_{(\xi,\epsilon)}^{\frac{2}{n-2}} w\psi\big) = \int_{\mathbb{R}^n_+} f\psi + \int_{\partial\mathbb{R}^n_+} \bar{f}\psi \tag{2.10}$$

for all $\psi \in \Sigma_{(\xi,\epsilon)}$. Moreover, if we set $\mathcal{G}_{(\xi,\epsilon)}(f,\bar{f}) = w$, there exists $C = C(n) > 0$ such that

$$\big\|\mathcal{G}_{(\xi,\epsilon)}(f,\bar{f})\big\|_{\Sigma} \leqslant C\|f\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n_+)} + C\|\bar{f}\|_{L^{\frac{2(n-1)}{n}}(\partial\mathbb{R}^n_+)}.$$

Sketch of the proof. Proceeding as in Proposition 2 in [6] and using Lemmas 2.1 and 2.2, we can prove that there exists $\theta = \theta(n) > 0$ such that

$$\int_{\mathbb{R}^n_+} |dw|^2 - n \int_{\partial\mathbb{R}^n_+} u_{(\xi,\epsilon)}^{\frac{2}{n-2}} w^2 \geqslant 2\theta |w|_\Sigma^2 - \frac{4}{\theta}\bigg(\int_{\partial\mathbb{R}^n_+} u_{(\xi,\epsilon)}^{\frac{n}{n-2}} w \bigg)^2$$

for all $w \in \Sigma_{(\xi,\epsilon)}$ and any pair $(\xi,\epsilon) \in \mathbb{R}^{n-1} \times (0,\infty)$. As in Corollary 3 in [6], we can show that there exists $0 < \alpha_0 = \alpha_0(n) \leqslant 1$ such that, whenever $|h(x)| + |\partial h(x)| + |\partial^2 h(x)| \leqslant \alpha_0$ for all $x \in \mathbb{R}^n_+$, we have

$$\int_{\mathbb{R}^n_+} \big(|dw|_g^2 + c_n R_g w^2\big) + \int_{\partial\mathbb{R}^n_+} \big(d_n \kappa_g w^2 - n u_{(\xi,\epsilon)}^{\frac{2}{n-2}} w^2\big) \geqslant \frac{\theta}{2} \|w\|_\Sigma^2 - \frac{1}{\theta} A(w)^2 \tag{2.11}$$

for all $w \in \Sigma_{(\xi,\epsilon)}$ and any pair $(\xi,\epsilon) \in \mathbb{R}^{n-1} \times (0,\infty)$. Here,

$$A(w) = \int_{\mathbb{R}^n_+} (\Delta_g u_{(\xi,\epsilon)} - c_n R_g u_{(\xi,\epsilon)}) w + \int_{\partial\mathbb{R}^n_+} \big(-d_n \kappa_g u_{(\xi,\epsilon)} + 2u^{\frac{n}{n-2}}\big) w.$$

In order to prove the existence part, we define the functional

$$T(w) = \int_{\mathbb{R}^n_+} \big(|dw|_g^2 + c_n R_g w^2 - 2fw\big) + \int_{\partial\mathbb{R}^n_+} \big(d_n \kappa_g w^2 - n u_{(\xi,\epsilon)}^{\frac{2}{n-2}} w^2 - 2\bar{f} w\big) + \frac{1}{\theta} A(w)^2$$

for $w \in \Sigma_{(\xi,\epsilon)}$ and use the estimate (2.11) to find a minimizer w_0 for T over all functions in $\Sigma_{(\xi,\epsilon)}$. The uniqueness part also uses (2.11). $\square$

Now, the following proposition is an application of the contraction principle using Proposition 2.4.

Proposition 2.5. *Let α_0 be the constant obtained in Proposition 2.4. There is a constant $\alpha_1 = \alpha_1(n)$, $0 < \alpha_1 \leqslant \alpha_0$, with the following property: if $|h(x)| + |\partial h(x)| + |\partial^2 h(x)| \leqslant \alpha_1$ for all $x \in \mathbb{R}^n_+$, given any pair $(\xi,\epsilon) \in \mathbb{R}^{n-1} \times (0,\infty)$ there exists a unique $v_{(\xi,\epsilon)} \in \Sigma$ such that $v_{(\xi,\epsilon)} - u_{(\xi,\epsilon)} \in \Sigma_{(\xi,\epsilon)}$ and*

$$\int_{\mathbb{R}^n_+} \big(\langle dv_{(\xi,\epsilon)}, d\psi\rangle_g + c_n R_g v_{(\xi,\epsilon)} \psi\big) + \int_{\partial\mathbb{R}^n_+} \big(d_n \kappa_g v_{(\xi,\epsilon)} \psi - (n-2)|v_{(\xi,\epsilon)}|^{\frac{2}{n-2}} v_{(\xi,\epsilon)} \psi\big) = 0$$

for all $\psi \in \Sigma_{(\xi,\epsilon)}$. Moreover, there exists $C = C(n) > 0$ such that

$$\|v_{(\xi,\epsilon)} - u_{(\xi,\epsilon)}\|_\Sigma \leqslant C\|\Delta_g u_{(\xi,\epsilon)} - c_n R_g u_{(\xi,\epsilon)}\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n_+)} + C\|d_n \kappa_g u_{(\xi,\epsilon)}\|_{L^{\frac{2(n-1)}{n}}(\partial\mathbb{R}^n_+)}. \tag{2.12}$$

In particular, $v_{(\xi,\epsilon)} \not\equiv 0$.

Observe that $v_{(\xi,\epsilon)}$ cannot be identically zero because of (2.12) and Proposition 2.3 with $\alpha = \alpha_1$ small.

Given a pair $(\xi,\epsilon) \in \mathbb{R}^{n-1} \times (0,\infty)$ we define the energy function

$$\mathcal{F}_g(\xi,\epsilon)=\int\limits_{\mathbb{R}^n_+}\left(|dv_{(\xi,\epsilon)}|_g^2+c_nR_gv_{(\xi,\epsilon)}^2\right)+\int\limits_{\partial\mathbb{R}^n_+}d_n\kappa_gv_{(\xi,\epsilon)}^2$$
$$-\frac{(n-2)^2}{n-1}\int\limits_{\partial\mathbb{R}^n_+}|v_{(\xi,\epsilon)}|^{\frac{2(n-1)}{n-2}}-\frac{n-2}{n-1}\int\limits_{\partial\mathbb{R}^n_+}u_{(\xi,\epsilon)}^{\frac{2(n-1)}{n-2}}. \tag{2.13}$$

Proposition 2.6. *Suppose that $|h(x)|+|\partial h(x)|+|\partial^2h(x)|\leqslant\alpha_1$ for all $x\in\mathbb{R}^n_+$, where α_1 is the constant obtained in Proposition* 2.5*. Choosing α_1 smaller if necessary, the function $\mathcal{F}_g$ is continuously differentiable and, if $(\bar\xi,\bar\epsilon)$ is a critical point of $\mathcal{F}_g$, then $v_{(\bar\xi,\bar\epsilon)}$ is a positive smooth solution of*

$$\begin{cases}\Delta_gv_{(\bar\xi,\bar\epsilon)}-c_nR_gv_{(\bar\xi,\bar\epsilon)}=0, & \text{in } \mathbb{R}^n_+,\\ \dfrac{\partial}{\partial x_n}v_{(\bar\xi,\bar\epsilon)}-d_n\kappa_gv_{(\bar\xi,\bar\epsilon)}+(n-2)v_{(\bar\xi,\bar\epsilon)}^{\frac{n}{n-2}}=0, & \text{on } \partial\mathbb{R}^n_+.\end{cases} \tag{2.14}$$

In the proof of Proposition 2.6 we use the following removable singularities theorem, which is a slight modification of Proposition 2.7 in [22]:

Lemma 2.7. *Let (M^n,g) be a Riemannian manifold with boundary ∂M. Let $x\in\partial M$ be a boundary point and $\mathcal{U}\subset M$ an open set containing x. Let u be a weak solution to*

$$\begin{cases}\Delta_gu+\phi u=0, & \text{in } \mathcal{U}\setminus\{x\},\\ \dfrac{\partial u}{\partial\eta}+\psi u=0, & \text{on } \mathcal{U}\cap\partial M\setminus\{x\},\end{cases}$$

where $\phi\in L^{\frac{n}{2}}(\mathcal{U})$ and $\psi\in L^{n-1}(\mathcal{U}\cap\partial M)$. Suppose that $u\in L^q(\mathcal{U})\cap L^p(\mathcal{U}\cap\partial M)$ for some $q>\frac{n}{n-2}$ and $p>\frac{n-1}{n-2}$. Then u is a weak solution to

$$\begin{cases}\Delta_gu+\phi u=0, & \text{in } \mathcal{U},\\ \dfrac{\partial u}{\partial\eta}+\psi u=0, & \text{on } \mathcal{U}\cap\partial M.\end{cases}$$

Sketch of the proof of Proposition 2.6. Given a pair $(\xi,\epsilon)\in\mathbb{R}^{n-1}\times(0,\infty)$, by the definition of $v_{(\xi,\epsilon)}$, there exist $b_a(\xi,\epsilon)\in\mathbb{R}$, $a=1,\dots,n$, such that

$$\int\limits_{\mathbb{R}^n_+}\left(\langle dv_{(\xi,\epsilon)},d\psi\rangle_g+c_nR_gv_{(\xi,\epsilon)}\psi\right)+\int\limits_{\partial\mathbb{R}^n_+}\left(d_n\kappa_gv_{(\xi,\epsilon)}\psi-(n-2)|v_{(\xi,\epsilon)}|^{\frac{2}{n-2}}v_{(\xi,\epsilon)}\psi\right)$$
$$=\sum_{a=1}^nb_a(\xi,\epsilon)\cdot\int\limits_{\partial\mathbb{R}^n_+}\phi_{(\xi,\epsilon,a)}\psi \tag{2.15}$$

for any $\psi\in\Sigma$. Following the same steps of Proposition 6 in [6] we can prove that $b_a(\bar\xi,\bar\epsilon)=0$ for $a=1,\dots,n$ and also that $v_{(\bar\xi,\bar\epsilon)}\geqslant0$ on $\partial\mathbb{R}^n_+$. In particular, Eq. (2.15) can be written as

$$\int\limits_{\mathbb{R}^n_+}\left(\langle dv_{(\bar\xi,\bar\epsilon)},d\psi\rangle_g+c_nR_gv_{(\bar\xi,\bar\epsilon)}\psi\right)+\int\limits_{\partial\mathbb{R}^n_+}\left(d_n\kappa_gv_{(\bar\xi,\bar\epsilon)}\psi-(n-2)v_{(\bar\xi,\bar\epsilon)}^{\frac{n}{n-2}}\psi\right)=0$$

for any $\psi\in\Sigma$. By a result of Cherrier in [10], $v_{(\bar\xi,\bar\epsilon)}$ is smooth.

The fact that $v_{(\bar{\xi},\bar{\epsilon})} > 0$ in $\mathbb{R}^n_+$ is just a consequence of the maximum principle, as follows. We set $\tilde{g} = \tilde{u}^{\frac{4}{n-2}} g$, where $\tilde{u}(x) = (1+|x|^2)^{\frac{2-n}{2}}$. Observe that $\tilde{u}$ satisfies $\Delta\tilde{u} + n(n-2)\tilde{u}^{\frac{n+2}{n-2}} = 0$ in $\mathbb{R}^n_+$ and we have

$$\begin{aligned} c_n R_{\tilde{g}} &= -\tilde{u}^{-\frac{n+2}{n-2}} \Delta\tilde{u} - \tilde{u}^{-\frac{n+2}{n-2}} (\Delta_g \tilde{u} - \Delta\tilde{u} - c_n R_g \tilde{u}) \\ &\geqslant n(n-2) - C\tilde{u}^{-\frac{n+2}{n-2}} \big\{ |h| \big|\partial^2\tilde{u}\big| + |\partial h||\partial\tilde{u}| + \big(\big|\partial^2 h\big| + |\partial h|^2\big)|\tilde{u}| \big\}. \end{aligned}$$

Using the facts that $h(x) = 0$ for $|x| \geqslant 1$ and $|h| + |\partial h| + |\partial^2 h| \leqslant C\alpha_1$ we can assume that $R_{\tilde{g}} > 0$, by choosing α_1 small.

Let S^n_+ be a hemisphere of $S^n_{1/2}$. We will use the well-known conformal equivalence between $S^n_+ \setminus \{x_0\}$ and $\mathbb{R}^n_+$ realized by the stereographic projection, where $x_0 \in \partial S^n_+$. Under this equivalence, the standard metric on S^n_+ is written on $\mathbb{R}^n_+$ as $\tilde{u}^{\frac{4}{n-2}}\delta$, where δ is the Euclidean metric on $\mathbb{R}^n_+$. We set $\tilde{v} = \tilde{u}^{-1} v_{(\bar{\xi},\bar{\epsilon})}$. By the properties (2.8) of the operators $L_g = \Delta_g - c_n R_g$ and $B_g = \frac{\partial}{\partial\eta} - d_n\kappa_g$, we have

$$L_{\tilde{g}}(\tilde{v}) = \tilde{u}^{-\frac{n+2}{n-2}} L_g v_{(\bar{\xi},\bar{\epsilon})} = 0, \quad \text{in } S^n_+,$$

and

$$B_{\tilde{g}}(\tilde{v}) + (n-2)\tilde{v}^{\frac{n}{n-2}} = \tilde{u}^{-\frac{n}{n-2}} B_g v_{(\bar{\xi},\bar{\epsilon})} + (n-2)\big(\tilde{u}^{-1} v_{(\bar{\xi},\bar{\epsilon})}\big)^{\frac{n}{n-2}} = 0, \quad \text{on } \partial S^n_+.$$

To establish the last two equations, we used Lemma 2.7.

Since $R_{\tilde{g}} > 0$, it follows from the maximum principle in S^n_+ and the Hopf Lemma that if $\tilde{v} \geqslant 0$ on ∂S^n_+ then we have either $\tilde{v} > 0$ or $\tilde{v} \equiv 0$ in S^n_+. The latter contradicts the last assertion of Proposition 2.5. Hence, $\tilde{v} \geqslant 0$ on ∂S^n_+ implies that $\tilde{v} > 0$ in S^n_+. Since we have proved that $v_{(\bar{\xi},\bar{\epsilon})} \geqslant 0$ on $\partial\mathbb{R}^n_+$, we conclude that $v_{(\bar{\xi},\bar{\epsilon})} > 0$ in $\mathbb{R}^n_+$. □

3. An estimate for the energy of a bubble

In this section we will show that the energy function $\mathcal{F}_g$ can be approximated by a certain auxiliary function.

We fix a multi-linear form $W : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ satisfying the algebraic properties of the Weyl tensor. We set

$$|W|^2 = \sum_{a,b,c,d=1}^{n} (W_{acbd} + W_{adbc})^2$$

and assume that $|W|^2 > 0$. Recall that throughout this article we work with indices $1 \leqslant i, j, k, l \leqslant n-1$ and $1 \leqslant a, b, c, d \leqslant n$ and set $\bar{x} = (x_1, \ldots, x_{n-1}, 0) \in \partial\mathbb{R}^n_+$ whenever $x = (x_1, \ldots, x_{n-1}, x_n) \in \mathbb{R}^n_+$. For $x \in \mathbb{R}^n_+$ we set

$$H_{ij}(x) = H_{ij}(\bar{x}) = W_{ikjl} x^k x^l \quad \text{and} \quad H_{nb}(x) = 0$$

and define $\bar{H}_{ab}(x) = f(|\bar{x}|^2) H_{ab}(x)$, where

$$f(s) = \sum_{j=0}^{d} a_j s^j. \tag{3.1}$$

The integer $0 < d < \frac{n-6}{4}$ and the coefficients $a_0, \dots, a_d \in \mathbb{R}$ will be chosen later. Observe that H is symmetric, trace-free, independent of the coordinate x_n and satisfies

$$x^a H_{ab}(x) = x^i H_{ib}(x) = 0 = \partial_a H_{ab}(x) = \partial_i H_{ib}(x), \quad \text{for any } x \in \mathbb{R}^n_+.$$

We define a Riemannian metric $g = \exp(h)$ on $\mathbb{R}^n_+$ where h is a trace-free symmetric two-tensor on $\mathbb{R}^n_+$ satisfying

$$\begin{cases} h_{ab}(x) = \mu\lambda^{2d} f\big(\lambda^{-2}|\bar{x}|^2\big) H_{ab}(x), & \text{for } |x| \leqslant \rho, \\ h_{ab}(x) = 0, & \text{for } |x| \geqslant 1. \end{cases}$$

Here, $\mu \leqslant 1$, $\lambda \leqslant \rho \leqslant 1$ and we suppose that $h_{nb}(x) = 0$ for any $x \in \mathbb{R}^n_+$ and $\partial_n h_{ab}(x) = 0$ for any $x \in \partial\mathbb{R}^n_+$. Observe that $h_{ab}(x) = O(\mu(\lambda + |x|)^{2d+2})$. We also assume that $|h| + |\partial h| + |\partial^2 h| \leqslant \alpha_1$ where α_1 is the constant obtained in Proposition 2.5. The second fundamental form of $\partial\mathbb{R}^n_+$ satisfies

$$\pi_{ij} = \Gamma^n_{ij} = \frac{1}{2}(g_{in,j} + g_{jn,i} - g_{ij,n}) = 0.$$

Using Proposition 2.5, for each pair $(\xi, \epsilon) \in \mathbb{R}^{n-1} \times (0, \infty)$ we choose $v_{(\xi,\epsilon)}$ to be the unique element of Σ such that $v_{(\xi,\epsilon)} - u_{(\xi,\epsilon)} \in \Sigma_{(\xi,\epsilon)}$ and

$$\int_{\mathbb{R}^n_+} \big(\langle dv_{(\xi,\epsilon)}, d\psi\rangle_g + c_n R_g v_{(\xi,\epsilon)}\psi\big) - (n-2)\int_{\partial\mathbb{R}^n_+} |v_{(\xi,\epsilon)}|^{\frac{2}{n-2}} v_{(\xi,\epsilon)}\psi = 0$$

for all $\psi \in \Sigma_{(\xi,\epsilon)}$.

We define $\Omega = \{(\xi, \epsilon) \in \mathbb{R}^{n-1} \times (0, \infty);\ |\xi| < 1,\ \frac{1}{2} < \epsilon < 2\}$. Similarly to Proposition 7 and Corollary 8 of [6] and Proposition 5 and Corollary 6 of [8] we have the estimates

$$\|\Delta_g u_{(\xi,\epsilon)} - c_n R_g u_{(\xi,\epsilon)}\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n_+)} \leqslant C\mu\lambda^{2d+2} + C\left(\frac{\lambda}{\rho}\right)^{\frac{n-2}{2}},$$

$$\begin{aligned} &\big\|\Delta_g u_{(\xi,\epsilon)} - c_n R_g u_{(\xi,\epsilon)} + \mu\lambda^{2d} f\big(\lambda^{-2}|\bar{x}|^2\big) H_{ij}\partial_i\partial_j u_{(\xi,\epsilon)}\big\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n_+)} \\ &\quad \leqslant C\mu^2\lambda^{4d+4} + C\left(\frac{\lambda}{\rho}\right)^{\frac{n-2}{2}} \end{aligned} \tag{3.2}$$

and

$$\|v_{(\xi,\epsilon)} - u_{(\xi,\epsilon)}\|_{L^{\frac{2n}{n-2}}(\mathbb{R}^n_+)} + \|v_{(\xi,\epsilon)} - u_{(\xi,\epsilon)}\|_{L^{\frac{2(n-1)}{n-2}}(\partial\mathbb{R}^n_+)} \leqslant C\mu\lambda^{2d+2} + C\left(\frac{\lambda}{\rho}\right)^{\frac{n-2}{2}} \tag{3.3}$$

for any pair $(\xi, \epsilon) \in \lambda\Omega$.

In order to refine the estimate (3.3), using Proposition 2.4 with $h_{ab} = 0$ we choose the function $w_{(\xi,\epsilon)}$ to be the unique element of $\Sigma_{(\xi,\epsilon)}$ satisfying

$$\int_{\mathbb{R}^n_+} \langle dw_{(\xi,\epsilon)}, d\psi\rangle - \int_{\partial\mathbb{R}^n_+} n u_{(\xi,\epsilon)}^{\frac{2}{n-2}} w_{(\xi,\epsilon)}\psi = -\int_{\mathbb{R}^n_+} \mu\lambda^{2d} f\big(\lambda^{-2}|\bar{x}|^2\big) H_{ij}\partial_i\partial_j u_{(\xi,\epsilon)}\psi \tag{3.4}$$

for all $\psi \in \Sigma_{(\xi,\epsilon)}$. Observe that, since $x^i H_{ij}(x) = 0$ for any $x \in \mathbb{R}^n_+$, we have $w_{(0,\epsilon)} = 0$.

Proposition 3.1. *The function $w_{(\xi,\epsilon)}$ is smooth and satisfies, for any $(\xi,\epsilon)\in\lambda\Omega$,*

$$\left|\partial^k w_{(\xi,\epsilon)}(x)\right| \leqslant C\lambda^{\frac{n-2}{2}}\mu\left(\lambda+|x|\right)^{2d+4-k-n}, \quad \textit{for all } x\in\mathbb{R}^n_+,\ k=0,1,2,$$

and

$$\begin{aligned}&\|v_{(\xi,\epsilon)}-u_{(\xi,\epsilon)}-w_{(\xi,\epsilon)}\|_{L^{\frac{2n}{n-2}}(\mathbb{R}^n_+)}+\|v_{(\xi,\epsilon)}-u_{(\xi,\epsilon)}-w_{(\xi,\epsilon)}\|_{L^{\frac{2(n-1)}{n-2}}(\partial\mathbb{R}^n_+)}\\ &\quad\leqslant C\mu^{\frac{n}{n-2}}\lambda^{\frac{(2d+2)\cdot n}{n-2}}+C\left(\frac{\lambda}{\rho}\right)^{\frac{n-2}{2}}. \end{aligned} \tag{3.5}$$

Proof. First observe that there exist real numbers $b_a(\xi,\epsilon)$, $1\leqslant a\leqslant n$, such that $w_{(\xi,\epsilon)}$ satisfies

$$\begin{aligned}&\int_{\mathbb{R}^n_+}\langle dw_{(\xi,\epsilon)},d\psi\rangle-\int_{\partial\mathbb{R}^n_+}nu_{(\xi,\epsilon)}^{\frac{2}{n-2}}w_{(\xi,\epsilon)}\psi\\ &\quad=-\int_{\mathbb{R}^n_+}\mu\lambda^{2d}f\left(\lambda^{-2}|\bar{x}|^2\right)H_{ij}\partial_i\partial_j u_{(\xi,\epsilon)}\psi+\sum_{a=1}^n b_a(\xi,\epsilon)\int_{\partial\mathbb{R}^n_+}\phi_{(a,\xi,\epsilon)}\psi \end{aligned} \tag{3.6}$$

for all $\psi\in\Sigma$. Hence, it follows from standard elliptic theory that $w_{(\xi,\epsilon)}$ is smooth.

Now we are going to prove the pointwise estimates. Observe that

$$\left\|\mu\lambda^{2d}f\left(\lambda^{-2}|\bar{x}|^2\right)H_{ij}(x)\partial_i\partial_j u_{(\xi,\epsilon)}(x)\right\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n_+)}\leqslant C\mu\lambda^{2d+2}. \tag{3.7}$$

Then we apply Proposition 2.4 with $h_{ab}=0$ and use the estimates (2.3) and (3.7) to conclude that

$$\|w_{(\xi,\epsilon)}\|_{L^{\frac{2n}{n-2}}(\mathbb{R}^n_+)}+\|w_{(\xi,\epsilon)}\|_{L^{\frac{2(n-1)}{n-2}}(\partial\mathbb{R}^n_+)}\leqslant C'\|w_{(\xi,\epsilon)}\|_{\Sigma}\leqslant C\mu\lambda^{2d+2}.$$

Moreover, we can use Eq. (3.6) with $\psi=\phi_{(\xi,\epsilon,a)}$ to conclude that

$$\sum_{a=0}^n\left|b_a(\xi,\epsilon)\right|\leqslant C\mu\lambda^{2d+2}.$$

Hence,

$$\left|\Delta w_{(\xi,\epsilon)}(x)\right|=\left|\mu\lambda^{2d}f\left(\lambda^{-2}|\bar{x}|^2\right)H_{ij}(x)\partial_i\partial_j u_{(\xi,\epsilon)}(x)\right|\leqslant\mu\lambda^{\frac{n-2}{2}}\left(\lambda+|x|\right)^{2d+2-n},$$

for all $x\in\mathbb{R}^n_+$, and

$$\left|\frac{\partial}{\partial x_n}w_{(\xi,\epsilon)}(x)+nu_{(\xi,\epsilon)}^{\frac{2}{n-2}}w_{(\xi,\epsilon)}(x)\right|=\left|-\sum_{a=1}^n b_a(\xi,\epsilon)\phi_{(a,\xi,\epsilon)}(x)\right|\leqslant\mu\lambda^{\frac{n}{2}}\left(\lambda+|x|\right)^{2d+2-n}$$

for all $x\in\partial\mathbb{R}^n_+$.

Claim. $\sup_{x\in\mathbb{R}^n_+}(\lambda+|x|)^{\frac{n-2}{2}}|w_{(\xi,\epsilon)}(x)|\leqslant C\mu\lambda^{2d+2}$.

We fix $x_0 \in \mathbb{R}^n_+$ and set $r = \frac{1}{2}(\lambda + |x_0|)$. Then we see that

$$u_{(\xi,\epsilon)}^{\frac{2}{n-2}}(x) \leqslant Cr^{-1}, \quad \text{for all } x \in B_r^+(x_0),$$

$$\left|\frac{\partial}{\partial x_n} w_{(\xi,\epsilon)}(x) + n u_{(\xi,\epsilon)}^{\frac{2}{n-2}} w_{(\xi,\epsilon)}(x)\right| \leqslant C\mu\lambda^{\frac{n}{2}} r^{2d+2-n}, \quad \text{for all } x \in B_r^+(x_0) \cap \partial\mathbb{R}^n_+,$$

and

$$\left|\Delta w_{(\xi,\epsilon)}(x)\right| \leqslant C\mu\lambda^{\frac{n-2}{2}} r^{2d+2-n}, \quad \text{for all } x \in B_r^+(x_0).$$

It follows from standard interior estimates that

$$\begin{aligned}
r^{\frac{n-2}{2}} \left|w_{(\xi,\epsilon)}(x_0)\right| &\leqslant C\|w_{(\xi,\epsilon)}\|_{L^{\frac{2n}{n-2}}(B_r^+(x_0))} + Cr^{\frac{n+2}{2}} \|\Delta w_{(\xi,\epsilon)}\|_{L^\infty(B_r^+(x_0))} \\
&\quad + Cr^{\frac{n}{2}} \left\| \frac{\partial}{\partial x_n} w_{(\xi,\epsilon)} + n u_{(\xi,\epsilon)}^{\frac{2}{n-2}} w_{(\xi,\epsilon)} \right\|_{L^\infty(B_r^+(x_0)\cap\partial\mathbb{R}^n_+)} \\
&\leqslant C\mu\lambda^{2d+2} + C\mu\lambda^{\frac{n-2}{2}} r^{2d+2+\frac{2-n}{2}} + C\mu\lambda^{\frac{n}{2}} r^{2d+2-\frac{n}{2}} \\
&\leqslant C\mu\lambda^{2d+2},
\end{aligned}$$

since we are assuming that $d < \frac{n-6}{4}$. This proves the claim.

Since $\sup_{x\in\mathbb{R}^n_+} |x|^{\frac{n-2}{2}} |w_{(\xi,\epsilon)}(x)| < \infty$, for all $x = (x_1, \dots, x_{n-1}, x_n) \in \mathbb{R}^n_+$ we have

$$\begin{aligned}
w_{(\xi,\epsilon)}(x) = &-\frac{1}{(n-2)\sigma_{n-2}} \int_{\mathbb{R}^n_+} \left(|x-y|^{2-n} + |\tilde{x}-y|^{2-n}\right) \Delta w_{(\xi,\epsilon)}(y)\, dy \\
&- \frac{1}{(n-2)\sigma_{n-2}} \int_{\partial\mathbb{R}^n_+} \left(|x-y|^{2-n} + |\tilde{x}-y|^{2-n}\right) \frac{\partial}{\partial y_n} w_{(\xi,\epsilon)}(y)\, dy,
\end{aligned}$$

where $\tilde{x} = (x_1, \dots, x_{n-1}, -x_n)$. Now we use a bootstrap argument to prove the pointwise estimates. It follows from the last two inequalities that

$$\begin{aligned}
&\sup_{x\in\mathbb{R}^n_+} \left(\lambda + |x|\right)^{\beta} \left|w_{(\xi,\epsilon)}(x)\right| \\
&\quad \leqslant C \sup_{x\in\mathbb{R}^n_+} \left(\lambda + |x|\right)^{\beta+2} \left|\Delta w_{(\xi,\epsilon)}(x)\right| + C \sup_{x\in\partial\mathbb{R}^n_+} \left(\lambda + |x|\right)^{\beta+1} \left|\frac{\partial}{\partial x_n} w_{(\xi,\epsilon)}(x)\right|
\end{aligned}$$

for all $0 < \beta < n-2$. Since

$$\left|\Delta w_{(\xi,\epsilon)}(x)\right| \leqslant \mu\lambda^{\frac{n-2}{2}} \left(\lambda + |x|\right)^{2d+2-n}, \quad \text{for all } x \in \mathbb{R}^n_+,$$

and

$$\left|\frac{\partial}{\partial x_n} w_{(\xi,\epsilon)}(x)\right| \leqslant n u_{(\xi,\epsilon)}^{\frac{2}{n-2}}(x) \left|w_{(\xi,\epsilon)}(x)\right| + \mu\lambda^{\frac{n}{2}} \left(\lambda + |x|\right)^{2d+2-n}, \quad \text{for all } x \in \partial\mathbb{R}^n_+,$$

we see that

$$\sup_{x\in\mathbb{R}^n_+}\big(\lambda+|x|\big)^{\beta}\big|w_{(\xi,\epsilon)}(x)\big| \leqslant C\lambda \sup_{x\in\partial\mathbb{R}^n_+}\big(\lambda+|x|\big)^{\beta-1}\big|w_{(\xi,\epsilon)}(x)\big| + C\mu\lambda^{\beta+2d+3-\frac{n}{2}}$$

for all $0<\beta\leqslant n-4-2d$. Integrating we obtain

$$\sup_{x\in\mathbb{R}^n_+}\big(\lambda+|x|\big)^{n-2d-4}\big|w_{(\xi,\epsilon)}(x)\big| \leqslant C\mu\lambda^{\frac{n-2}{2}}.$$

The derivative estimates follow from elliptic theory and the estimate (3.5) is analogous to Corollary 10 of [6] and Corollary 8 of [8]. $\square$

In the next proposition we estimate the energy $\mathcal{F}_g$.

Proposition 3.2. *Let $\mathcal{F}_g$ be the function defined by the formula* (2.13)*. For any pair $(\xi,\epsilon)\in\lambda\Omega$ we have the estimate*

$$\begin{aligned}&\bigg|\mathcal{F}_g(\xi,\epsilon)-\frac{1}{2}\int\limits_{B^+_\rho(0)} h_{il}h_{jl}\partial_i u_{(\xi,\epsilon)}\partial_j u_{(\xi,\epsilon)}+\frac{c_n}{4}\int\limits_{B^+_\rho(0)}(\partial_l h_{ij})^2u^2_{(\xi,\epsilon)}\\ &\quad-\int\limits_{\mathbb{R}^n_+}\mu\lambda^{2d}f\big(\lambda^{-2}|\bar{x}|^2\big)H_{ij}\partial_i\partial_j u_{(\xi,\epsilon)}w_{(\xi,\epsilon)}\bigg|\\ &\quad\leqslant C\mu^{\frac{2(n-1)}{n-2}}\lambda^{\frac{(4d+4)(n-1)}{n-2}}+C\mu\lambda^{2d+2}\left(\frac{\lambda}{\rho}\right)^{\frac{n-2}{2}}+C\left(\frac{\lambda}{\rho}\right)^{n-2}.\end{aligned}$$

Proof. It follows from the definition of $v_{(\xi,\epsilon)}$ that

$$\begin{aligned}&\int\limits_{\mathbb{R}^n_+}\big\{\big\langle dv_{(\xi,\epsilon)},d(v_{(\xi,\epsilon)}-u_{(\xi,\epsilon)})\big\rangle_g+c_nR_gv_{(\xi,\epsilon)}(v_{(\xi,\epsilon)}-u_{(\xi,\epsilon)})\big\}\\ &\quad-(n-2)\int\limits_{\partial\mathbb{R}^n_+}|v_{(\xi,\epsilon)}|^{\frac{2}{n-2}}v_{(\xi,\epsilon)}(v_{(\xi,\epsilon)}-u_{(\xi,\epsilon)})=0.\end{aligned}\tag{3.8}$$

We set

$$\begin{aligned}\varrho=&\int\limits_{\mathbb{R}^n_+}\big\{\big\langle du_{(\xi,\epsilon)},d(v_{(\xi,\epsilon)}-u_{(\xi,\epsilon)})\big\rangle_g+c_nR_gu_{(\xi,\epsilon)}(v_{(\xi,\epsilon)}-u_{(\xi,\epsilon)})\big\}\\ &-\int\limits_{\mathbb{R}^n_+}h_{ij}\partial_i\partial_j u_{(\xi,\epsilon)}(v_{(\xi,\epsilon)}-u_{(\xi,\epsilon)})-(n-2)\int\limits_{\partial\mathbb{R}^n_+}u^{\frac{n}{n-2}}_{(\xi,\epsilon)}(v_{(\xi,\epsilon)}-u_{(\xi,\epsilon)}).\end{aligned}\tag{3.9}$$

Summing (3.8) and (3.9), we obtain

$$\begin{aligned}
\varrho = &\int_{\mathbb{R}^n_+} \left\{ |dv_{(\xi,\epsilon)}|_g^2 + c_n R_g v_{(\xi,\epsilon)}^2 \right\} - \int_{\partial\mathbb{R}^n_+} \left\{ \frac{(n-2)^2}{n-1} |v_{(\xi,\epsilon)}|^{\frac{2(n-1)}{n-2}} + \frac{n-2}{n-1} u_{(\xi,\epsilon)}^{\frac{2(n-1)}{n-2}} \right\} \\
&- \int_{\partial\mathbb{R}^n_+} \left\{ \frac{n-2}{n-1} |v_{(\xi,\epsilon)}|^{\frac{2(n-1)}{n-2}} + \frac{(n-2)^2}{n-1} u_{(\xi,\epsilon)}^{\frac{2(n-1)}{n-2}} \right\} \\
&- (n-2) \int_{\partial\mathbb{R}^n_+} \left(u_{(\xi,\epsilon)}^{\frac{2}{n-2}} - |v_{(\xi,\epsilon)}|^{\frac{2}{n-2}} \right) u_{(\xi,\epsilon)} v_{(\xi,\epsilon)} + 2(n-2) \int_{\partial\mathbb{R}^n_+} u_{(\xi,\epsilon)}^{\frac{2(n-1)}{n-2}} \\
&- \int_{\mathbb{R}^n_+} \left\{ |du_{(\xi,\epsilon)}|_g^2 + c_n R_g u_{(\xi,\epsilon)}^2 + h_{ij} \partial_i \partial_j u_{(\xi,\epsilon)} \left(v_{(\xi,\epsilon)} - u_{(\xi,\epsilon)} \right) \right\}.
\end{aligned} \tag{3.10}$$

We set

$$B = \int_{\mathbb{R}^n_+} \left\{ |du_{(\xi,\epsilon)}|_g^2 - |du_{(\xi,\epsilon)}|^2 + c_n R_g u_{(\xi,\epsilon)}^2 + h_{ij} \partial_i \partial_j u_{(\xi,\epsilon)} \left(v_{(\xi,\epsilon)} - u_{(\xi,\epsilon)} \right) \right\}$$

and observe that $\int_{\mathbb{R}^n_+} |du_{(\xi,\epsilon)}|^2 = (n-2) \int_{\partial\mathbb{R}^n_+} u_{(\xi,\epsilon)}^{\frac{2(n-1)}{n-2}}$. Hence,

$$\begin{aligned}
\mathcal{F}_g(\xi,\epsilon) - B &= \frac{n-2}{n-1} \int_{\partial\mathbb{R}^n_+} \left\{ |v_{(\xi,\epsilon)}|^{\frac{2(n-1)}{n-2}} - u_{(\xi,\epsilon)}^{\frac{2(n-1)}{n-2}} \right\} \\
&\quad - (n-2) \int_{\partial\mathbb{R}^n_+} \left(|v_{(\xi,\epsilon)}|^{\frac{2}{n-2}} - u_{(\xi,\epsilon)}^{\frac{2}{n-2}} \right) u_{(\xi,\epsilon)} v_{(\xi,\epsilon)} + \varrho \\
&= O\left(\lambda^{\frac{(4d+4)(n-1)}{n-2}} \mu^{\frac{2(n-1)}{n-2}} + \left(\frac{\lambda}{\rho} \right)^{n-1} \right) + \varrho
\end{aligned} \tag{3.11}$$

where in the last inequality we used the estimate

$$\begin{aligned}
&\left| \int_{\partial\mathbb{R}^n_+} \left(|v_{(\xi,\epsilon)}|^{\frac{2}{n-2}} - u_{(\xi,\epsilon)}^{\frac{2}{n-2}} \right) u_{(\xi,\epsilon)} v_{(\xi,\epsilon)} - \frac{1}{n-1} \int_{\partial\mathbb{R}^n_+} \left(|v_{(\xi,\epsilon)}|^{\frac{2(n-1)}{n-2}} - u_{(\xi,\epsilon)}^{\frac{2(n-1)}{n-2}} \right) \right| \\
&\quad \leqslant C \mu^{\frac{2(n-1)}{n-2}} \lambda^{\frac{(4d+4)(n-1)}{n-2}} + C \left(\frac{\lambda}{\rho} \right)^{n-1},
\end{aligned}$$

which is similar to Proposition 12 in [6] and Proposition 10 in [8]. On the other hand,

$$\begin{aligned}
&\left| B - \frac{1}{2} \int_{B^+_\rho(0)} h_{il} h_{jl} \partial_i u_{(\xi,\epsilon)} \partial_j u_{(\xi,\epsilon)} + \frac{c_n}{4} \int_{B^+_\rho(0)} (\partial_l h_{ij})^2 u_{(\xi,\epsilon)}^2 - \int_{\mathbb{R}^n_+} \mu \lambda^{2d} f\left(\lambda^{-2} |\bar{x}|^2 \right) H_{ij} \partial_i \partial_j u_{(\xi,\epsilon)} w_{(\xi,\epsilon)} \right| \\
&\leqslant C \mu^{\frac{2(n-1)}{n-2}} \lambda^{\frac{(4d+4)(n-1)}{n-2}} + C \mu \lambda^{2d+2} \left(\frac{\lambda}{\rho} \right)^{\frac{n-2}{2}} + C \left(\frac{\lambda}{\rho} \right)^{n-2}.
\end{aligned} \tag{3.12}$$

Here, we are omitting the details of the last inequality, but we should mention that we are using the facts that $\partial_j h_{ij}(x)=0$, for $|x|\leqslant\rho$, and $\sum_{j=1}^{n-1}h_{jj}=0$ and the identity $u_{(\xi,\epsilon)}\partial_i\partial_j u_{(\xi,\epsilon)}-\frac{n}{n-2}\partial_i u_{(\xi,\epsilon)}\partial_j u_{(\xi,\epsilon)}=-\frac{1}{n-2}|du_{(\xi,\epsilon)}|^2\delta_{ij}$.

Next we are going to estimate ϱ using its definition (Eq. (3.9)). Integrating by parts and using the second equation of (2.1), we obtain

$$\begin{aligned}
|\varrho| &\leqslant \int_{\mathbb{R}^n_+}\left|-\Delta_g u_{(\xi,\epsilon)}(v_{(\xi,\epsilon)}-u_{(\xi,\epsilon)})+c_nR_g u_{(\xi,\epsilon)}(v_{(\xi,\epsilon)}-u_{(\xi,\epsilon)})-h_{ij}\partial_i\partial_j u_{(\xi,\epsilon)}(v_{(\xi,\epsilon)}-u_{(\xi,\epsilon)})\right|\\
&\leqslant \|\Delta_g u_{(\xi,\epsilon)}-c_nR_g u_{(\xi,\epsilon)}+h_{ij}\partial_i\partial_j u_{(\xi,\epsilon)}\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n_+)}\|v_{(\xi,\epsilon)}-u_{(\xi,\epsilon)}\|_{L^{\frac{2n}{n-2}}(\mathbb{R}^n_+)}\\
&\leqslant C\mu^3\lambda^{6d+6}+C\mu\lambda^{2d+2}\left(\frac{\lambda}{\rho}\right)^{\frac{n-2}{2}}+C\left(\frac{\lambda}{\rho}\right)^{n-2}.
\end{aligned}\tag{3.13}$$

Here, we used the estimates (3.2) and (3.3) in the last inequality.

The result now follows from (3.11), (3.12) and (3.13). □

4. Finding a critical point of an auxiliary function

Let us follow the notations of the last section. We define

$$F(\xi,\epsilon)=\frac{1}{2}\int_{\mathbb{R}^n_+}\bar H_{il}\bar H_{jl}\partial_i u_{(\xi,\epsilon)}\partial_j u_{(\xi,\epsilon)}-\frac{c_n}{4}\int_{\mathbb{R}^n_+}(\partial_l\bar H_{ij})^2u^2_{(\xi,\epsilon)}+\int_{\mathbb{R}^n_+}\bar H_{ij}\partial_i\partial_j u_{(\xi,\epsilon)}z_{(\xi,\epsilon)}$$

where $z_{(\xi,\epsilon)}$ is the unique element of $\Sigma_{(\xi,\epsilon)}$ that satisfies

$$\int_{\mathbb{R}^n_+}\langle dz_{(\xi,\epsilon)},d\psi\rangle-\int_{\partial\mathbb{R}^n_+}nu^{\frac{2}{n-2}}_{(\xi,\epsilon)}z_{(\xi,\epsilon)}\psi=-\int_{\mathbb{R}^n_+}\bar H_{ij}\partial_i\partial_j u_{(\xi,\epsilon)}\psi \tag{4.1}$$

for any $\psi\in\Sigma_{(\xi,\epsilon)}$. The function $z_{(\xi,\epsilon)}$ is obtained in Proposition 2.4 with $h_{ab}=0$.

In this section we will show that the function $F(\xi,\epsilon)$ has a critical point, which is a strict local minimum. Recall that throughout this article we use indices $1\leqslant i,i,j,k,l,m,p,q,r\leqslant n-1$.

Since $\bar H_{ab}(-x)=\bar H_{ab}(x)$ for any $x\in\mathbb{R}^n_+$, the function $F(\xi,\epsilon)$ satisfies $F(\xi,\epsilon)=F(-\xi,\epsilon)$ for all $(\xi,\epsilon)\in\mathbb{R}^{n-1}\times(0,\infty)$. In particular,

$$\frac{\partial}{\partial\xi_p}F(0,\epsilon)=\frac{\partial^2}{\partial\epsilon\partial\xi_p}F(0,\epsilon)=0,\quad\text{for all }\epsilon>0. \tag{4.2}$$

Proposition 4.1. *We have*

$$\begin{aligned}
\int_{S^{n-2}_r}(\partial_l\bar H_{ij})^2(x)x^px^q=&\frac{2\sigma_{n-2}r^{n+2}}{(n-1)(n+1)(n+3)}(W_{ipjl}+W_{iljp})(W_{iqjl}+W_{iljq})\\
&\cdot\left\{(n+3)f\left(r^2\right)^2+8r^2f\left(r^2\right)f'\left(r^2\right)+4r^4f'\left(r^2\right)^2\right\}\\
&+\frac{\sigma_{n-2}r^{n+2}}{(n-1)(n+1)(n+3)}(W_{ikjl}+W_{iljk})^2\delta_{pq}\\
&\cdot\left\{(n+3)f\left(r^2\right)^2+4r^2f\left(r^2\right)f'\left(r^2\right)+2r^4f'\left(r^2\right)^2\right\}.
\end{aligned}$$

Proof. Analogous to Proposition 15 in [8]. □

Corollary 4.2. *We have*

$$\int_{S_r^{n-2}} (\partial_l \bar{H}_{ij})^2(x) = \frac{\sigma_{n-2} r^n}{(n-1)(n+1)} (W_{ikjl} + W_{iljk})^2 \big\{(n+1) f\big(r^2\big)^2 + 4r^2 f\big(r^2\big) f'\big(r^2\big) + 2r^4 f'\big(r^2\big)^2\big\}.$$

As a consequence we can prove the next result.

Proposition 4.3. *We have*

$$F(0,\epsilon) = -\frac{c_n \cdot \sigma_{n-2}}{4(n-1)(n+1)} (W_{ikjl} + W_{iljk})^2 \int_0^\infty \int_0^\infty r^n \big\{(n+1) f\big(r^2\big)^2 + 4r^2 f\big(r^2\big) f'\big(r^2\big) + 2r^4 f'\big(r^2\big)^2\big\}$$
$$\cdot \epsilon^{n-2} \big((\epsilon+t)^2 + r^2\big)^{2-n} \, dr \, dt.$$

Proof. It follows from symmetry arguments that $z_{(0,\epsilon)} = 0$ and

$$\int_{S_r^{n-2}} \bar{H}_{il} \bar{H}_{jl} \partial_i u_{(0,\epsilon)} \partial_j u_{(0,\epsilon)}(x)$$
$$= \int_{S_r^{n-2}} \frac{(n-2)^2 \epsilon^{n-2}}{((\epsilon + x_n)^2 + |\bar{x}|^2)^n} f\big(|\bar{x}|^2\big)^2 W_{iplq} W_{jrlm} x^i x^j x^p x^q x^r x^m = 0.$$

Hence, we have

$$F(0,\epsilon) = -\frac{c_n}{4} \int_{\mathbb{R}^n_+} (\partial_l \bar{H}_{ij})^2(x) u^2_{(0,\epsilon)}(x)$$
$$= -\frac{c_n}{4} \int_0^\infty \int_0^\infty \int_{S_r^{n-2}} (\partial_l \bar{H}_{ij})^2(x) u^2_{(0,\epsilon)}(x) \, d\sigma_r(x) \, dr \, dx_n.$$

The result now follows from Corollary 4.2. □

We write

$$F(0,\epsilon) = -\beta_n \cdot \sum_{q=0}^{2d} \alpha_q \int_0^\infty \int_0^\infty r^{2q+n} \epsilon^{n-2} \big((\epsilon+t)^2 + r^2\big)^{2-n} \, dr \, dt,$$

where

$$\beta_n = \frac{c_n \cdot \sigma_{n-2}}{4(n-1)(n+1)} (W_{ikjl} + W_{iljk})^2,$$

and define the coefficients $\alpha_q \in \mathbb{R}$ by the formula

$$\sum_{q=0}^{2d} \alpha_q s^q = (n+1) f(s)^2 + 4sf(s)f'(s) + 2s^2 f'(s)^2. \tag{4.3}$$

Here, d is the integer in the formula (3.1). Changing variables $t' = t/\epsilon$ and $r' = r/\epsilon$ we obtain

$$F(0,\epsilon) = -\beta_n \cdot \sum_{q=0}^{2d} \alpha_q \epsilon^{2q+4} \int_0^\infty \int_0^\infty \frac{r^{2q+n}}{((1+t)^2 + r^2)^{n-2}} \, dr \, dt$$

and, changing variables $r' = r/(1+t)$,

$$F(0,\epsilon) = -\beta_n \cdot \sum_{q=0}^{2d} \alpha_q \epsilon^{2q+4} \int_0^\infty \frac{1}{(1+t)^{n-5-2q}} \, dt \cdot \int_0^\infty \frac{r^{2q+n}}{(1+r^2)^{n-2}} \, dr.$$

Now, we have

$$\int_0^\infty \frac{1}{(1+t)^{n-5-2q}} \, dt = \frac{1}{n-6-2q}$$

and

$$\int_0^\infty \frac{r^{2q+n}}{(1+r^2)^{n-2}} \, dr = \left\{ \prod_{j=0}^{q} \frac{n-1+2j}{n-5-2j} \right\} \cdot \int_0^\infty \frac{r^{n-2}}{(1+r^2)^{n-2}} \, dr,$$

where we used the fact that

$$\int_0^\infty \frac{s^\alpha \, ds}{(1+s^2)^m} = \frac{2m-\alpha-3}{\alpha+1} \int_0^\infty \frac{s^{\alpha+2} \, ds}{(1+s^2)^m}, \quad \text{for } \alpha + 3 < 2m. \tag{4.4}$$

Hence, we can write

$$F(0,\epsilon) = -\beta_n \cdot I\big(\epsilon^2\big) \cdot \int_{r=0}^\infty \frac{r^{n-2}}{(1+r^2)^{n-2}} \, dr \tag{4.5}$$

where

$$I(s) = \sum_{q=0}^{2d} \frac{\alpha_q}{n-6-2q} \left\{ \prod_{j=0}^{q} \frac{n-1+2j}{n-5-2j} \right\} s^{q+2}. \tag{4.6}$$

We will now turn our attention to the second order derivatives of the function $F(\xi,\epsilon)$.

Proposition 4.4. *We have*

$$
\begin{aligned}
\frac{\partial^2}{\partial\xi_p\partial\xi_q}F(0,\epsilon) = & -\frac{2(n-2)^2\sigma_{n-2}}{(n-1)(n+1)(n+3)}(W_{ipjl}+W_{iljp})(W_{iqjl}+W_{iljq}) \\
& \cdot \int_0^\infty\int_0^\infty \frac{\epsilon^{n-2}}{((\epsilon+x_n)^2+r^2)^n} r^{n+4}\big(2f\big(r^2\big)f'\big(r^2\big)+r^2f'\big(r^2\big)^2\big)\,dr\,dt \\
& -\frac{(n-2)^2\sigma_{n-2}}{2(n-1)(n+1)(n+3)}(W_{ikjl}+W_{iljk})^2\delta_{pq} \\
& \cdot \int_0^\infty\int_0^\infty \frac{\epsilon^{n-2}}{((\epsilon+x_n)^2+r^2)^n} r^{n+4}\big\{2f\big(r^2\big)f'\big(r^2\big)+r^2f'\big(r^2\big)^2\big\}\,dr\,dt \\
& +\frac{(n-2)^2\sigma_{n-2}}{4(n-1)^2(n+1)}(W_{ikjl}+W_{iljk})^2\delta_{pq} \\
& \cdot \int_0^\infty\int_0^\infty \frac{\epsilon^{n-2}}{((\epsilon+x_n)^2+r^2)^{n-1}} r^{n+4}f'\big(r^2\big)^2\,dr\,dt.
\end{aligned}
\tag{4.7}
$$

Proof. As in Proposition 21 in [6] we can prove the identity

$$
\begin{aligned}
\frac{\partial^2}{\partial\xi_p\partial\xi_q}F(0,\epsilon) = {} & (n-2)^2\int_{\mathbb{R}^n_+}\frac{\epsilon^{n-2}}{((\epsilon+x_n)^2+|\bar{x}|^2)^n}\bar{H}_{pl}(x)\bar{H}_{ql}(x) \\
& -\frac{(n-2)^2}{4}\int_{\mathbb{R}^n_+}\frac{\epsilon^{n-2}}{((\epsilon+x_n)^2+|\bar{x}|^2)^n}\big(\partial_l\bar{H}_{ij}(x)\big)^2x^px^q \\
& +\frac{(n-2)^2}{8(n-1)}\int_{\mathbb{R}^n_+}\frac{\epsilon^{n-2}}{((\epsilon+x_n)^2+|\bar{x}|^2)^{n-1}}\big(\partial_l\bar{H}_{ij}(x)\big)^2\delta_{pq}.
\end{aligned}
$$

Then the assertion follows from a computation similar to Proposition 20 in [8]. □

Let us define constants β_q, for $q=0,\dots,2d-1$, by the following expression:

$$
\sum_{q=0}^{2d-1}\beta_q s^q = 2f(s)f'(s)+sf'(s)^2.
$$

Proposition 4.5. *We have*

$$
\begin{aligned}
& \int_0^\infty\int_0^\infty \frac{\epsilon^{n-2}}{((\epsilon+x_n)^2+r^2)^n} r^{n+4}\big(2f\big(r^2\big)f'\big(r^2\big)+r^2f'\big(r^2\big)^2\big)\,dr\,dt \\
& \quad = J\big(\epsilon^2\big)\cdot\int_0^\infty\frac{r^{n+2}}{(1+r^2)^n}\,dr,
\end{aligned}
\tag{4.8}
$$

where

$$J(s)=\sum_{q=0}^{2d-1}\frac{\beta_q s^{q+2}}{n-6-2q}\cdot\left\{\prod_{j=0}^{q}\frac{n+3+2j}{n-5-2j}\right\}.$$

Proof.

$$\begin{aligned}
&\int_0^\infty\int_0^\infty\frac{\epsilon^{n-2}}{((\epsilon+x_n)^2+r^2)^n}r^{n+4}\big(2f\big(r^2\big)f'\big(r^2\big)+r^2f'\big(r^2\big)^2\big)\,dr\,dt\\
&\quad=\sum_{q=0}^{2d-1}\beta_q\int_0^\infty\int_0^\infty\frac{\epsilon^{n-2}r^{n+4+2q}}{((\epsilon+x_n)^2+r^2)^n}\,dr\,dt\\
&\quad=\sum_{q=0}^{2d-1}\beta_q\epsilon^{2q+4}\int_0^\infty\int_0^\infty\frac{r^{n+4+2q}}{((1+x_n)^2+r^2)^n}\,dr\,dt\\
&\quad=\sum_{q=0}^{2d-1}\beta_q\epsilon^{2q+4}\int_0^\infty\frac{1}{(1+t)^{n-5-2q}}\,dt\int_0^\infty\frac{r^{n+4+2q}}{(1+r^2)^n}\,dr.
\end{aligned}$$

Now we observe that

$$\int_0^\infty\frac{1}{(1+t)^{n-5-2q}}\,dt=\frac{1}{n-6-2q}$$

and apply the formula (4.4) to see that

$$\int_0^\infty\frac{r^{n+4+2q}}{(1+r^2)^n}\,dr=\left\{\prod_{j=0}^{q}\frac{n+3+2j}{n-5-2j}\right\}\cdot\int_0^\infty\frac{r^{n+2}}{(1+r^2)^n}\,dr.\qquad\square$$

4.1. *The case $n\geqslant 53$*

In this case we choose $d=1$ in Eq. (3.1). Then the coefficients α_q in Eq. (4.3) are given by

$$\alpha_0=(n+1)a_0^2,\qquad \alpha_1=2(n+3)a_0a_1,\qquad \alpha_2=(n+7)a_1^2.$$

Thus, derivating $I(s)$ in the expression (4.6) we obtain

$$\begin{aligned}
I'(s)&=\sum_{q=0}^{2}\frac{(q+2)\alpha_q}{n-6-2q}\left\{\prod_{j=0}^{q}\frac{n-1+2j}{n-5-2j}\right\}s^{q+1}\\
&=\frac{2\alpha_0(n-1)}{(n-6)(n-5)}\cdot s+\frac{3\alpha_1(n-1)(n+1)}{(n-8)(n-5)(n-7)}\cdot s^2+\frac{4\alpha_2(n-1)(n+1)(n+3)}{(n-10)(n-5)(n-7)(n-9)}\cdot s^3\\
&=\frac{2(n+1)(n-1)}{n-5}\left\{\frac{1}{n-6}a_0^2s+\frac{3(n+3)}{(n-8)(n-7)}a_0a_1s^2+\frac{2(n+3)(n+7)}{(n-10)(n-7)(n-9)}a_1^2s^3\right\}.
\end{aligned}$$

Now we choose $a_1 = -1$ and define the polynomial p_n by

$$p_n(a_0) = \frac{a_0^2}{n-6} - \frac{3(n+3)a_0}{(n-8)(n-7)} + \frac{2(n+3)(n+7)}{(n-10)(n-7)(n-9)}.$$

Hence,

$$I'(1) = \frac{2(n+1)(n-1)}{n-5} p_n(a_0).$$

The discriminant of p_n is then given by

$$\begin{aligned}\operatorname{discrim}(p_n) &= \frac{(n+3)^2}{(n-7)^2(n-8)^2}\left\{9 - \frac{8(n-7)(n-8)^2(n+7)}{(n+3)(n-6)(n-9)(n-10)}\right\}\\ &= \frac{(n+3)^2}{(n-7)^2(n-8)^2}\,\frac{q(n)}{(n+3)(n-6)(n-9)(n-10)},\end{aligned}$$

where

$$q(n) = 9(n+3)(n-6)(n-9)(n-10) - 8(n-7)(n-8)^2(n+7).$$

Observe that

$$q'(n) = 4n^3 - 210n^2 + 2082n - 5624$$

and

$$q''(n) = 6\big(2n^2 - 70n + 347\big).$$

Since the roots $\frac{70\pm\sqrt{2124}}{4}$ of q'' are less than 53, we see that $q''(n) > 0$ for $n \geqslant 53$. Since $q(53) = 105\,696$ and $q'(53) = 110\,340$, we conclude that $\operatorname{discrim}(p_n) > 0$ for $n \geqslant 53$. Hence, if we set

$$a_0 = \frac{(n+3)(n-6)}{2(n-7)(n-8)}\left\{3 + \sqrt{9 - \frac{8(n-7)(n-8)^2(n+7)}{(n+3)(n-6)(n-9)(n-10)}}\right\},$$

then $s = 1$ is critical point of $I(s)$. According to Proposition A.1 in Appendix A, $I''(1) < 0$ for $n \geqslant 53$.

Now we will handle $J(s)$, as defined in Proposition 4.5. We have

$$J(s) = \frac{(n+3)\beta_0 s^2}{(n-6)(n-5)} + \frac{(n+3)(n+5)\beta_1 s^3}{(n-8)(n-5)(n-7)}$$

where

$$\beta_0 = 2a_0a_1 \quad \text{and} \quad \beta_1 = 3a_1^2.$$

Hence,

$$J(s) = \frac{(n+3)a_1}{n-5}\left\{\frac{2a_0 s^2}{n-6} + \frac{3(n+5)a_1 s^3}{(n-8)(n-7)}\right\}.$$

If we set a_0 and a_1 as above we have

$$J(1) = \frac{n+3}{(n-8)(n-5)(n-7)} \cdot \left\{6 - (n+3)\sqrt{9 - \frac{8(n-7)(n-8)^2(n+7)}{(n+3)(n-6)(n-9)(n-10)}}\right\}.$$

According to Proposition A.2 in Appendix A, $J(1) < 0$ for $n \geqslant 53$.

From Eqs. (4.2), (4.5), (4.7) and (4.8) and the above results we can conclude the following:

Proposition 4.6. *Suppose that $n \geqslant 53$. If we set $a_1 = -1$ and*

$$a_0 = \frac{(n+3)(n-6)}{2(n-7)(n-8)}\left\{3 + \sqrt{9 - \frac{8(n-7)(n-8)^2(n+7)}{(n+3)(n-6)(n-9)(n-10)}}\right\},$$

then $I'(1) = 0$, $I''(1) < 0$ and $J(1) < 0$. In particular, the function $F(\xi, \epsilon)$ has a strict local minimum at the point $(0, 1)$.

4.2. The case $25 \leqslant n \leqslant 52$

In this case we choose $d = 4$ in Eq. (3.1). The coefficients α_q in Eq. (4.3) are then given by

$$\begin{aligned}
\alpha_0 &= (n+1)a_0^2,\\
\alpha_1 &= 2(n+3)a_0a_1,\\
\alpha_2 &= 2(n+5)a_0a_2 + (n+7)a_1^2,\\
\alpha_3 &= 2(n+11)a_1a_2 + 2(n+7)a_0a_3,\\
\alpha_4 &= 2(n+15)a_1a_3 + (n+17)a_2^2 + 2(n+9)a_0a_4,\\
\alpha_5 &= 2(n+23)a_2a_3 + 2(n+19)a_1a_4,\\
\alpha_6 &= (n+31)a_3^2 + 2(n+29)a_2a_4,\\
\alpha_7 &= 2(n+39)a_3a_4,\\
\alpha_8 &= (n+49)a_4^2.
\end{aligned}$$

Thus, derivating $I(s)$ in the expression (4.6) we obtain

$$I'(s) = \sum_{q=0}^{8} \frac{(q+2)\alpha_q}{n-6-2q}\left\{\prod_{j=0}^{q} \frac{n-1+2j}{n-5-2j}\right\} s^{q+1}.$$

Now we choose $a_1 = -3/5$, $a_2 = 1/8$, $a_3 = -1/125$, $a_4 = 10^{-4}$ and define the polynomial r_n by $r_n(a_0) = I'(1)$. Hence,

$$\begin{aligned}
r_n(a_0) = {} & \frac{2(n-1)(n+1)}{(n-6)(n-5)} \cdot a_0^2 + \left\{\sum_{q=1}^{4} \gamma_q(n) \frac{q+2}{n-6-2q} \prod_{j=0}^{q} \frac{n-1+2j}{n-5-2j}\right\} \cdot a_0\\
& + \sum_{q=2}^{8} \delta_q(n) \frac{q+2}{n-6-2q} \prod_{j=0}^{q} \frac{n-1+2j}{n-5-2j},
\end{aligned}$$

where

$$\gamma_1(n) = -\frac{6}{5}(n+3), \qquad \gamma_2(n) = \frac{n+5}{4}, \qquad \gamma_3(n) = -\frac{2}{125}(n+7), \qquad \gamma_4(n) = \frac{n+9}{5000},$$
$$\delta_2(n) = \frac{9(n+7)}{25}, \qquad \delta_3(n) = -\frac{3(n+11)}{20}, \qquad \delta_4(n) = \frac{1009n + 16\,385}{40\,000},$$
$$\delta_5(n) = -\frac{53n + 1207}{25\,000}, \qquad \delta_6(n) = \frac{89n + 2709}{10^6}, \qquad \delta_7(n) = -\frac{n+39}{625\,000}, \qquad \delta_8(n) = \frac{n+49}{10^8}.$$

Direct computations show that $\mathrm{discrim}(r_n) > 0$ for $25 \leqslant n \leqslant 52$.

If we choose

$$a_0 = \frac{(n-6)(n-5)}{4(n-1)(n+1)} \cdot \left\{ -\sum_{q=1}^{4} \gamma_q(n) \frac{q+2}{n-6-2q} \prod_{j=0}^{q} \frac{n-1+2j}{n-5-2j} + \sqrt{\mathrm{discrim}(r_n)} \right\}$$

then $s = 1$ is critical point of $I(s)$. For $25 \leqslant n \leqslant 52$, direct computations show that $I''(1)$ is of the form $-e_1 - e_2\sqrt{e_3}$, where e_1, e_2, e_3 are positive rational numbers.

The function $J(s)$, defined in Proposition 4.5, is written as

$$J(s) = \sum_{q=0}^{7} \frac{\beta_q s^{q+2}}{n-6-2q} \cdot \left\{ \prod_{j=0}^{q} \frac{n+3+2j}{n-5-2j} \right\},$$

where

$$\beta_0 = 2a_0a_1, \qquad \beta_1 = 4a_0a_2 + 3a_1^2, \qquad \beta_2 = 6a_0a_3 + 10a_1a_2, \qquad \beta_3 = 8a_0a_4 + 14a_1a_3 + 8a_2^2,$$
$$\beta_4 = 18a_1a_4 + 22a_2a_3, \qquad \beta_5 = 28a_2a_4 + 15a_3^2, \qquad \beta_6 = 38a_3a_4, \qquad \beta_7 = 24a_4^2.$$

For $25 \leqslant n \leqslant 52$, direct computations show that $J(1)$ is of the form $-e_1 - e_2\sqrt{e_3}$, where e_1, e_2, e_3 are positive rational numbers. From Eqs. (4.2), (4.5), (4.7) and (4.8) and the above results we can conclude the following:

Proposition 4.7. *Suppose that* $25 \leqslant n \leqslant 52$. *If* $a_1 = -3/5$, $a_2 = 1/8$, $a_3 = -1/125$, $a_4 = 10^{-4}$ *and*

$$a_0 = \frac{(n-6)(n-5)}{4(n-1)(n+1)} \cdot \left\{ -\sum_{q=1}^{4} \gamma_q(n) \frac{q+2}{n-6-2q} \prod_{j=0}^{q} \frac{n-1+2j}{n-5-2j} + \sqrt{\mathrm{discrim}(r_n)} \right\}$$

then $I'(1) = 0$, $I''(1) < 0$ *and* $J(1) < 0$. *In particular, the function* $F(\xi, \epsilon)$ *has a strict local minimum at the point* $(0, 1)$.

5. Proof of the Main Theorem

In this section we will make use of the two-tensor H, defined on $\mathbb{R}^n_+$, the polynomial f and the open set $\Omega \subset \mathbb{R}^{n-1} \times (0, \infty)$, which were defined in Section 3. As in Sections 4.1 and 4.2, we fix $d = 1$ if $n \geqslant 53$ and $d = 4$ if $25 \leqslant n \leqslant 52$. We set $D_r(0) = \{x \in \partial\mathbb{R}^n_+;\ |x| < r\}$.

The basic ingredient in the proof of the Main Theorem is the following result:

Proposition 5.1. *Assume that $n \geqslant 25$. Let g be a smooth Riemannian metric on $\mathbb{R}^n_+$ expressed as $g = \exp(h)$, where h is a symmetric trace-free two-tensor on $\mathbb{R}^n_+$ satisfying the following properties*:

$$\begin{cases} h_{ab}(x) = \mu\lambda^{2d} f\big(\lambda^{-2}|\bar{x}|\big) H_{ab}(x), & \text{for } |x| \leqslant \rho, \\ h_{ab}(x) = 0, & \text{for } |x| \geqslant 1, \\ h_{nb}(x) = 0, & \text{for } x \in \mathbb{R}^n_+, \\ \partial_n h_{ab}(x) = 0, & \text{for } x \in \partial\mathbb{R}^n_+, \end{cases} \tag{5.1}$$

where $a, b = 1, \dots, n$. We also assume that

$$\big|h(x)\big| + \big|\partial h(x)\big| + \big|\partial^2 h(x)\big| \leqslant \alpha \leqslant \alpha_1, \quad \text{for all } x \in \mathbb{R}^n_+,$$

where α_1 is the constant obtained in Proposition 2.5.

If α and $\mu^{-2}\lambda^{n-4d-6}\rho^{2-n}$ are sufficiently small, then there exists a positive smooth function v satisfying

$$\begin{cases} \Delta_g v - c_n R_g v = 0, & \text{in } \mathbb{R}^n_+, \\ \dfrac{\partial}{\partial x_n} v - d_n \kappa_g v + (n-2) v^{\frac{n}{n-2}} = 0, & \text{on } \partial\mathbb{R}^n_+, \end{cases} \tag{5.2}$$

and

$$\int_{\partial\mathbb{R}^n_+} v^{\frac{2(n-1)}{n-2}} < \left(\frac{Q(B^n, \partial B)}{n-2}\right)^{n-1}. \tag{5.3}$$

Moreover, there exists $c = c(n) > 0$ such that

$$\sup_{D_\lambda(0)} v \geqslant c\lambda^{\frac{2-n}{2}}. \tag{5.4}$$

Proof. It follows from the fact that

$$(n+1) f(s)^2 + 4s f(s) f'(s) + 2s^2 f'(s)^2 = (n-1) f(s)^2 + 2\big(f(s) + s f'(s)\big)^2$$

and Proposition 4.3 that $F(0,1) < 0$. According to Propositions 4.6 and 4.7, we can choose the coefficients $a_0, \dots, a_d$ in the formula (3.1) such that the point $(0,1)$ is a strict local minimum of F. Hence, we can find an open set $\Omega' \subset \Omega$ such that $(0,1) \in \Omega'$ and

$$F(0,1) < \inf_{(\xi,\epsilon) \in \partial\Omega'} F(\xi,\epsilon) < 0.$$

Observe that $u_{(\lambda\xi,\lambda\epsilon)}(\lambda x) = \lambda^{-\frac{n-2}{2}} u_{(\xi,\epsilon)}(x)$ and $w_{(\lambda\xi,\lambda\epsilon)}(\lambda x) = \mu\lambda^{2d+2-\frac{n-2}{2}} z_{(\xi,\epsilon)}(x)$ for all $x \in \mathbb{R}^n_+$. Here, $w_{(\xi,\epsilon)}$ and $z_{(\xi,\epsilon)}$ are the functions defined by the formulas (3.4) and (4.1) respectively. Thus, it follows from Proposition 3.2 that

$$\big|\mathcal{F}_g(\lambda\xi, \lambda\epsilon) - \mu^2\lambda^{4d+4} F(\xi,\epsilon)\big| \leqslant C\mu^{\frac{2(n-1)}{n-2}} \lambda^{\frac{(4d+4)(n-1)}{n-2}} + C\mu\lambda^{2d+2}\left(\frac{\lambda}{\rho}\right)^{\frac{n-2}{2}} + C\left(\frac{\lambda}{\rho}\right)^{n-2}$$

for all $(\xi,\epsilon) \in \Omega$. Hence,

$$\begin{aligned}&\left|\mu^{-2}\lambda^{-4d-4}\mathcal{F}_g(\lambda\xi,\lambda\epsilon)-F(\xi,\epsilon)\right|\\&\quad\leqslant C\mu^{\frac{2}{n-2}}\lambda^{\frac{4d+4}{n-2}}+C\mu^{-1}\lambda^{\frac{n-4d-6}{2}}\rho^{\frac{2-n}{2}}+C\mu^{-2}\lambda^{n-4d-6}\rho^{2-n}\end{aligned}$$

for all $(\xi,\epsilon)\in\Omega$. If $\mu^{-2}\lambda^{n-4d-6}\rho^{2-n}$ is sufficiently small then we have

$$\mathcal{F}_g(0,\lambda)<\inf_{(\xi,\epsilon)\in\partial\Omega'}\mathcal{F}_g(\lambda\xi,\lambda\epsilon)<0.$$

Thus we conclude that there exists a point $(\bar{\xi},\bar{\epsilon})\in\Omega'$ such that

$$\mathcal{F}_g(\lambda\bar{\xi},\lambda\bar{\epsilon})=\inf_{(\xi,\epsilon)\in\Omega'}\mathcal{F}_g(\lambda\xi,\lambda\epsilon)<0.$$

By Proposition 2.6, the function $v=v_{(\lambda\bar{\xi},\lambda\bar{\epsilon})}$ obtained in Proposition 2.5 is a positive smooth solution to Eqs. (5.2). Hence, by the definition of $\mathcal{F}_g$ (see the formula (2.13)) and the formula (2.2), we have

$$\frac{n-2}{n-1}\int_{\partial\mathbb{R}^n_+}v^{\frac{2(n-1)}{n-2}}=\frac{n-2}{n-1}\left(\frac{Q(B^n,\partial B)}{n-2}\right)^{n-1}+\mathcal{F}(\lambda\bar{\xi},\lambda\bar{\epsilon}).$$

This implies the inequality (5.3).

In order to prove the inequality (5.4), observe that

$$\|v-u_{(\lambda\bar{\xi},\lambda\bar{\epsilon})}\|_{L^{\frac{2(n-1)}{n-2}}(D_\lambda(0))}\leqslant\|v-u_{(\lambda\bar{\xi},\lambda\bar{\epsilon})}\|_{L^{\frac{2(n-1)}{n-2}}(\partial\mathbb{R}^n_+)}\leqslant C\alpha$$

by Propositions 2.3 and 2.5. Hence,

$$\left|D_\lambda(0)\right|^{\frac{n-2}{2(n-1)}}\sup_{D_\lambda(0)}v\geqslant\|v\|_{L^{\frac{2(n-1)}{n-2}}(D_\lambda(0))}\geqslant-C\alpha+\|u_{(\lambda\bar{\xi},\lambda\bar{\epsilon})}\|_{L^{\frac{2(n-1)}{n-2}}(D_\lambda(0))}.$$

Now, the inequality (5.4) follows from choosing α sufficiently small. □

Now the Main Theorem follows from the next theorem, using the conformal equivalence between $B^n\setminus\{(0,\dots,0,-1)\}$ and $\mathbb{R}^n_+$ (see Lemma 2.1), the properties (2.8) and Lemma 2.7.

Theorem 5.2. *Assume that $n\geqslant 25$. Then there exists a smooth Riemannian metric g on $\mathbb{R}^n_+$ with the following properties*:

(a) $g_{ab}(x)=\delta_{ab}$ *for* $|x|\geqslant 1/2$;
(b) *g is not conformally flat*;
(c) *$\partial\mathbb{R}^n_+$ is totally geodesic with respect to the induced metric by g*;
(d) *there exists a sequence of positive smooth functions $\{v_\nu\}_{\nu=1}^\infty$ satisfying*

$$\begin{cases}\Delta_g v_\nu-c_nR_gv_\nu=0, & \text{in } \mathbb{R}^n_+,\\ \dfrac{\partial}{\partial x_n}v_\nu-d_n\kappa_gv_\nu+(n-2)v_\nu^{\frac{n}{n-2}}=0, & \text{on } \partial\mathbb{R}^n_+,\end{cases}\tag{5.5}$$

for all ν,

$$\int_{\partial\mathbb{R}^n_+}v_\nu^{\frac{2(n-1)}{n-2}}<\left(\frac{Q(B^n,\partial B)}{n-2}\right)^{n-1},$$

for all ν, and $\sup_{D_1(0)}v_\nu\to\infty$ as $\nu\to\infty$.

Proof. Let $\chi : \mathbb{R} \to \mathbb{R}$ be a smooth cutoff function such that $\chi(t) = 1$ for $t \leqslant 1$ and $\chi(t) = 0$ for $t \geqslant 2$. We define the trace-free symmetric two-tensor h on $\mathbb{R}^n_+$ by

$$h_{ab}(x) = \sum_{N=N_0}^{\infty} \chi\big(4N^2|x - x_N|\big)2^{-dN} f\big(2^N|\bar{x} - x_N|\big) H_{ab}(x - x_N)$$

where $x_N = (\frac{1}{N}, 0, \dots, 0) \in \partial\mathbb{R}^n_+$. Observe that h is smooth and satisfies $h_{an}(x) = 0$ for $x \in \mathbb{R}^n_+$ and $\partial_n h_{ab}(x) = 0$ for $x \in \partial\mathbb{R}^n_+$. If N_0 is sufficiently large, then $h_{ab}(x) = 0$ for $|x| \geqslant \frac{1}{2}$ and $|h(x)| + |\partial h(x)| + |\partial^2 h(x)| \leqslant \alpha$ for $x \in \mathbb{R}^n_+$, with α sufficiently small as in Proposition 5.1. Then we define the metric $g(x) = \exp(h(x))$ for $x \in \mathbb{R}^n_+$ and the result follows from Proposition 5.1. $\square$

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Appendix A

In this section we establish some results used in Section 4.1. The notations here are the same of that section. In particular, we fix $a_1 = -1$ and

$$a_0 = \frac{(n+3)(n-6)}{2(n-7)(n-8)}\left\{3 + \sqrt{9 - \frac{8(n-7)(n-8)^2(n+7)}{(n+3)(n-6)(n-9)(n-10)}}\right\}.$$

Proposition A.1. *We have $I''(1) < 0$ for $n \geqslant 53$.*

Proof. If $25 \leqslant n \leqslant 69$, direct computations show that $I''(1)$ is of the form $-e_1 - e_2\sqrt{e_3}$, where e_1, e_2, e_3 are positive rational numbers. We are going to prove that $I''(1) < 0$ for $n \geqslant 70$. We write

$$a_0 = \frac{(n+3)(n-6)}{2(n-7)(n-8)}\left\{3 + \sqrt{9 - \frac{8p_A(n)}{p_B(n)}}\right\},$$

where $p_A(n) = (n-7)(n-8)^2(n+7)$, $p_B(n) = (n+3)(n-6)(n-9)(n-10)$ and define

$$q_L(n) = p_A(n) - p_B(n) \quad \text{and} \quad q_U(n) = \alpha p_B(n) - p_A(n),$$

where $\alpha = \frac{31\,439}{28\,800}$.

Claim. *$q_L(n) > 0$ for $n \geqslant 9$ and $q_U(n) > 0$ for $n \geqslant 70$.*

In order to prove the claim, first observe that the forth order terms of q_L cancel out and we have $q_L(n) = 6n^3 - 114n^2 + 712n - 1516$. Hence, $q''_L(n) = 36n - 228 > 0$ for $n \geqslant 7$, $q_L(9) = 32$ and $q'_L(9) = 118$. Thus, $q_L(n) > 0$ for $n \geqslant 9$.

Now we observe that

$$q_U(n) = \frac{2639}{28\,800}n^4 - \frac{115\,429}{14\,400}n^3 + \frac{1\,207\,877}{9600}n^2 - \frac{282\,161}{400}n + \frac{218\,809}{160}.$$

Hence, $q'''_U(n) = \frac{2639}{1200}n - \frac{115\,439}{2400} > 0$ for $n \geqslant 70$, $q_U(70) = \frac{287\,074}{15}$, $q'_U(70) = \frac{178\,522\,037}{7200}$ and $q''_U(70) = \frac{10\,910\,017}{4800}$. Thus, $q_U(n) > 0$ for $n \geqslant 70$, proving the claim.

We assume that $n \geqslant 70$. In particular, we conclude from the claim that $\alpha > \frac{p_A(n)}{p_B(n)} > 1$, which implies

$$\frac{2(n+3)(n-6)}{(n-7)(n-8)} > a_0 > \frac{(n+3)(n-6)}{2(n-7)(n-8)}\bigl(3+\sqrt{9-8\alpha}\bigr).$$

Now we use this estimate in

$$I''(1) = \frac{2(n+1)(n-1)}{n-5}\left\{\frac{a_0^2}{n-6} - \frac{6(n+3)a_0}{(n-8)(n-7)} + \frac{6(n+3)(n+7)}{(n-10)(n-7)(n-9)}\right\}$$

to see that

$$\frac{(n-5)I''(1)}{2(n+1)(n-1)} < \frac{4(n+3)^3(n-6)}{(n-7)^2(n-8)^2} - \frac{3(3+\sqrt{9-8\alpha})(n+3)^2(n-6)}{(n-7)^2(n-8)^2} + \frac{6(n+3)(n+7)}{(n-10)(n-7)(n-9)}.$$

This can be written as

$$I''(1) < \frac{2(n+3)(n+1)(n-1)\gamma(n)}{(n-8)^2(n-10)(n-5)(n-7)^2(n-9)},$$

where

$$\gamma(n) = -(5+3\sqrt{9-8\alpha})(n+3)(n-6)(n-10)(n-9) + 6(n+7)(n-7)(n-8)^2.$$

In order to complete our proof, we will show that $\gamma(n) < 0$ under our assumption on the dimension. Observe that $\gamma(n) = -\frac{11}{20}n^4 + \frac{481}{10}n^3 - \frac{15\,099}{20}n^2 + \frac{21\,162}{5}n - 8205$. Hence $\gamma'''(n) = -\frac{66}{5}n + \frac{1443}{5} < 0$ for $n \geqslant 70$, $\gamma(70) = -118\,392$, $\gamma'(70) = -\frac{744\,953}{5}$ and $\gamma''(70) = -\frac{136\,479}{10}$. Now the result follows. $\square$

Proposition A.2. *We have $J(1) < 0$ for $n \geqslant 53$.*

Proof. Let us assume that $n \geqslant 53$. We want to show that $(n+3)\sqrt{9-\frac{8p_A(n)}{p_B(n)}} - 6 > 0$, where we are using the polynomials p_A and p_B as in the proof of Proposition A.1. We set again $q_U(n) = \alpha p_B(n) - p_A(n)$ and choose $\alpha = \frac{7047}{6272}$.

Claim. $q_U(n) > 0$.

In order to prove the claim, first observe that

$$q_U(n) = \frac{775}{6272}n^4 - \frac{27\,341}{3136}n^3 + \frac{814\,983}{6272}n^2 - \frac{551\,233}{784}n + \frac{2\,063\,213}{1568}.$$

Hence, $q_U'''(n) = \frac{2325}{784}n - \frac{82\,023}{1568} > 0$ for $n \geqslant 53$, $q_U(53) = \frac{169\,857}{28}$, $q_U'(53) = \frac{20\,672\,955}{1568}$ and $q_U''(53) = \frac{5\,182\,395}{3136}$. Thus, $q_U(n) > 0$ for $n \geqslant 53$, proving the claim.

The claim implies that $(n+3)\sqrt{9-\frac{8p_A(n)}{p_B(n)}} > (n+3)\sqrt{9-8\alpha}$, which reduces the problem to prove that

$$(n+3)\sqrt{9-8\alpha} - 6 \geqslant 0. \tag{A.1}$$

On the other hand, the fact that $\alpha = \frac{1}{8}\{9 - \frac{36}{56^2}\}$ implies $\frac{1}{8}\{9 - \frac{36}{(n+3)^2}\} \geqslant \alpha$, which is equivalent to the inequality (A.1). $\square$

References

[1] S. Almaraz, A compactness theorem for scalar-flat metrics on manifolds with boundary, Calc. Var. Partial Differential Equations, doi:10.1007/s00526-010-0365-8, in press.
[2] S. Almaraz, An existence theorem of conformal scalar-flat metrics on manifolds with boundary, Pacific J. Math. 248 (1) (2010) 1–22.
[3] A. Ambrosetti, Y. Li, A. Malchiodi, On the Yamabe problem and the scalar curvature problem under boundary conditions, Math. Ann. 322 (4) (2002) 667–699.
[4] T. Aubin, Équations différentielles non linéaires et probléme de Yamabe concernant la courbure scalaire, J. Math. Pures Appl. 55 (1976) 269–296.
[5] M. Berti, A. Malchiodi, Non-compactness and multiplicity results for the Yamabe problem on S^n, J. Funct. Anal. 180 (1) (2001) 210–241.
[6] S. Brendle, Blow-up phenomena for the Yamabe equation, J. Amer. Math. Soc. 21 (4) (2008) 951–979.
[7] S. Brendle, S. Chen, An existence theorem for the Yamabe problem on manifolds with boundary, arXiv:0908.4327v2.
[8] S. Brendle, F. Marques, Blow-up phenomena for the Yamabe equation II, J. Differential Geom. 81 (2009) 225–250.
[9] S. Chen, Conformal deformation to scalar flat metrics with constant mean curvature on the boundary in higher dimensions, arXiv:0912.1302v2.
[10] P. Cherrier, Problèmes de Neumann non linéaires sur les variétés Riemannienes, J. Funct. Anal. 57 (1984) 154–206.
[11] Z. Djadli, A. Malchiodi, M. Ould Ahmedou, Prescribing scalar and boundary mean curvature on the three dimensional half sphere, J. Geom. Anal. 13 (2) (2003) 255–289.
[12] Z. Djadli, A. Malchiodi, M. Ould Ahmedou, The prescribed boundary mean curvature problem on $\mathbb{B}^4$, J. Differential Equations 206 (2) (2004) 373–398.
[13] O. Druet, From one bubble to several bubbles: the low-dimensional case, J. Differential Geom. 63 (3) (2003) 399–473.
[14] O. Druet, Compactness for Yamabe metrics in low dimensions, Int. Math. Res. Not. IMRN 23 (2004) 1143–1191.
[15] J. Escobar, The Yamabe problem on manifolds with boundary, J. Differential Geom. 35 (1992) 21–84.
[16] J. Escobar, Conformal deformation of a Riemannian metric to a scalar flat metric with constant mean curvature on the boundary, Ann. of Math. 136 (1992) 1–50.
[17] J. Escobar, Conformal metrics with prescribed mean curvature on the boundary, Calc. Var. Partial Differential Equations 4 (1996) 559–592.
[18] V. Felli, M. Ould Ahmedou, Compactness results in conformal deformations of Riemannian metrics on manifolds with boundaries, Math. Z. 244 (2003) 175–210.
[19] V. Felli, M. Ould Ahmedou, A geometric equation with critical nonlinearity on the boundary, Pacific J. Math. 218 (1) (2005) 75–99.
[20] Z. Han, Y. Li, The Yamabe problem on manifolds with boundary: existence and compactness results, Duke Math. J. 99 (3) (1999) 489–542.
[21] M. Khuri, F. Marques, R. Schoen, A compactness theorem for the Yamabe problem, J. Differential Geom. 81 (1) (2009) 143–196.
[22] J. Lee, T. Parker, The Yamabe problem, Bull. Amer. Math. Soc. 17 (1987) 37–91.
[23] Y. Li, M. Zhang, Compactness of solutions to the Yamabe problem, II, Calc. Var. Partial Differential Equations 24 (2005) 185–237.
[24] Y. Li, M. Zhang, Compactness of solutions to the Yamabe problem, III, J. Funct. Anal. 245 (2) (2007) 438–474.
[25] Y. Li, M. Zhu, Yamabe type equations on three dimensional Riemannian manifolds, Commun. Contemp. Math. 1 (1) (1999) 1–50.
[26] F. Marques, A priori estimates for the Yamabe problem in the non-locally conformally flat case, J. Differential Geom. 71 (2005) 315–346.
[27] F. Marques, Existence results for the Yamabe problem on manifolds with boundary, Indiana Univ. Math. J. 54 (6) (2005) 1599–1620.
[28] F. Marques, Conformal deformation to scalar flat metrics with constant mean curvature on the boundary, Comm. Anal. Geom. 15 (2) (2007) 381–405.
[29] M. Ould Ahmedou, On the prescribed scalar and zero mean curvature on 3-D manifolds with umbilic boundary, Adv. Nonlinear Stud. 6 (2006) 13–46.
[30] R. Schoen, Conformal deformation of a Riemannian metric to constant scalar curvature, J. Differential Geom. 20 (1984) 479–495.
[31] R. Schoen, On the number of constant scalar curvature metrics in a conformal class, in: H.B. Lawson, K. Tenenblat (Eds.), Differential Geometry: A Symposium in Honor of Manfredo Do Carmo, Wiley, 1991, pp. 311–320.
[32] R. Schoen, S.-T. Yau, Lectures on Differential Geometry, Conference Proceedings and Lecture Notes in Geometry and Topology, vol. 1, International Press, Cambridge, 1994.
[33] R. Schoen, D. Zhang, Prescribed scalar curvature on the n-sphere, Calc. Var. Partial Differential Equations 4 (1) (1996) 1–25.
[34] N. Trudinger, Remarks concerning the conformal deformation of a Riemannian structure on compact manifolds, Ann. Sc. Norm. Super. Pisa Cl. Sci. (3) 22 (1968) 265–274.
[35] H. Yamabe, On a deformation of Riemannian structures on compact manifolds, Osaka Math. J. 12 (1960) 21–37.